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Kato–Milne cohomology and polynomial forms

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ABSTRACT

Given a prime number p, a field F with $\text{char}(F) = p$ and a positive integer n, we study the class-preserving modifications of Kato–Milne classes of decomposable differential forms. These modifications demonstrate a natural connection between differential forms and p-regular forms. A p-regular form is defined to be a homogeneous polynomial form of degree p for which there is no nonzero point where all the order $p-1$ partial derivatives vanish simultaneously. We define a $\widetilde{C}_{p,m}$ field to be a field over which every p-regular form of dimension greater than p^m is isotropic. The main results are that for a $\widetilde{C}_{p,m}$ field F, the symbol length of $H_p^2(F)$ is bounded from above by $p^{m-1}-1$ and for any $n \geq \lceil (m-1)\log_2(p) \rceil + 1$, $H_p^{n+1}(F) = 0$.

1. Introduction

In this paper we study the connection between the Kato–Milne cohomology groups $H_p^{n+1}(F)$ over a field F with $\text{char}(F) = p$ for some prime integer p, and homogeneous polynomial forms of degree p over F. The three main objectives of this work are:

1. Finding a number n_0 such that for any $n \geq n_0$, $H_p^{n+1}(F) = 0$.
2. Finding an upper bound for the symbol length of $H_p^2(F)$, which in turn provides an upper bound for the symbol length of ${}_pBr(F)$.
3. Finding a number s such that any collection of s inseparably linked decomposable differential forms in $H_p^{n+1}(F)$ are also separably linked.

1.1. The Kato–Milne cohomology groups

Given a prime number p and a field F of $\text{char}(F) = p$, we consider the space of absolute differential forms Ω_F^1, which is defined to be the F-vector space generated by the symbols da subject to the relations

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$d(a+b) = da + db$ and $d(ab) = adb + bda$ for any $a, b \in F$. The space of n-differential forms Ω_F^n for any positive integer n is then defined by the n-fold exterior power $\Omega_F^n = \bigwedge^n(\Omega_F^1)$, which is consequently an F-vector space spanned by $da_1 \wedge \ldots \wedge da_n$, $a_i \in F$. The derivation d extends to an operator $d : \Omega_F^n \to \Omega_F^{n+1}$ by $d(a_0 da_1 \wedge \ldots \wedge da_n) = da_0 \wedge da_1 \wedge \ldots \wedge da_n$. We define $\Omega_F^0 = F$, $\Omega_F^n = 0$ for $n < 0$, and $\Omega_F = \bigoplus_{n \geq 0} \Omega_F^n$, the algebra of differential forms over F with multiplication naturally defined by

$$(a_0 da_1 \wedge \ldots \wedge da_n)(b_0 db_1 \wedge \ldots \wedge db_m) = a_0 b_0 da_1 \wedge \ldots \wedge da_n \wedge db_1 \wedge \ldots \wedge db_m\,.$$

There exists a well-defined group homomorphism $\Omega_F^n \to \Omega_F^n / d\Omega_F^{n-1}$, the Artin–Schreier map $\wp$, which acts on decomposable differential forms as follows:

$$\alpha \frac{d\beta_1}{\beta_1} \wedge \ldots \wedge \frac{d\beta_n}{\beta_n} \longmapsto (\alpha^p - \alpha) \frac{d\beta_1}{\beta_1} \wedge \ldots \wedge \frac{d\beta_n}{\beta_n}.$$

The group $H_p^{n+1}(F)$ is defined to be coker($\wp$). By [15], in the case of $p = 2$, there exists an isomorphism

$$H_2^{n+1}(F) \xrightarrow{\cong} I_q^{n+1}(F)/I_q^{n+2}(F), \text{ given by}$$
$$\alpha \frac{d\beta_1}{\beta_1} \wedge \ldots \wedge \frac{d\beta_n}{\beta_n} \longmapsto \langle\langle \beta_1, \ldots, \beta_n, \alpha]] \mod I_q^{n+2}(F)$$

where $\langle\langle \beta_1, \ldots, \beta_n, \alpha]]$ is a quadratic n-fold Pfister form.

By [14, Section 9.2], when $n = 1$, there exists an isomorphism

$$H_p^2(F) \xrightarrow{\sim} {}_pBr(F), \text{ given by}$$
$$\alpha \frac{d\beta}{\beta} \longmapsto [\alpha, \beta)_{p,F},$$

where $[\alpha, \beta)_{p,F}$ is the degree p cyclic p-algebra

$$F\langle x, y : x^p - x = \alpha, y^p = \beta, yxy^{-1} = x + 1\rangle.$$

In the special case of $p = 2$ and $n = 1$, these cyclic p-algebras are quaternion algebras $[\alpha, \beta)_{2,F}$ that can be identified with their norm forms which are quadratic 2-fold Pfister forms $\langle\langle \beta, \alpha]]$ (see [10, Corollary 12.2 (1)]).

1.2. C_m and $\widetilde{C}_{p,m}$ fields

A C_m field is a field F over which every homogeneous polynomial form of degree d in more than d^m variables is isotropic (i.e. has a nontrivial zero). It was suggested in [23, Chapter II, Section 4.5, Exercise 3 (b)] that if F is a C_m field with char(F) $\neq p$ then for any $n \geq m$, $H^{n+1}(F, \mu_p^{\otimes n}) = 0$. This fact is known for $p = 2$ because of the Milnor conjecture, proven in [24]. It was proven in [16] that for any prime $p > 3$, C_m field F with char(F) $\neq p$ and $n \geq \lceil (m-2)\log_2(p) + 1\rceil$, we have $H^{n+1}(F, \mu_p^{\otimes n}) = 0$. (The same result holds when $p = 3$ for $n \geq \lceil (m-3)\log_2(3) + 3\rceil$.) The analogous statement for fields F with char(F) $= p$ is that if F is a C_m field then $H_p^{n+1}(F) = 0$ for every $n \geq m$. This is true, as stated in [23, Chapter II, Section 4.5, Exercise 3 (a)] and proven explicitly in [2]. It follows from the fact that C_m fields F have p-rank at most m, i.e. $[F : F^p] \leq p^m$. We consider a somewhat different property of fields that avoids directly bounding their p-rank. We say that a homogeneous polynomial form of degree p over F is ***p*-regular** if there is no nonzero point where all the partial derivatives of order $p - 1$ vanish. We denote by $u_p(F)$ the maximal dimension of an anisotropic p-regular form over F. We say F is a $\widetilde{C}_{p,m}$ field if $u_p(F) \leq p^m$. We prove that if F is $\widetilde{C}_{p,m}$ then for any $n \geq \lceil (m-1)\log_2(p)\rceil + 1$, we have $H_p^{n+1}(F) = 0$. (See Section 2 for examples of $\widetilde{C}_{p,m}$ which are not C_m.)