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Nilpotence and generation in the stable module category

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ABSTRACT

Nilpotence has been studied in stable homotopy theory and algebraic geometry. We study the corresponding notion in modular representation theory of finite groups, and apply the discussion to the study of ghosts, and generation of the stable module category. In particular, we show that for a finitely generated kG-module M, the tensor M-generation number and the tensor M-ghost number are both equal to the degree of tensor nilpotence of a certain map associated with M.

1. Introduction

The study of nilpotence started in stable homotopy theory [11,13,20]. Analogues have also been studied in algebraic geometry, see for example Theorems 3.6 and 3.8 of Thomason [22]. In this paper, we study the analogous notion in modular representation theory, and relate it to tensor notions of ghost maps and categorical generation. The emphasis on concepts involving tensor closure seems practical in light of the fact that the thick tensor ideal generated by the trivial module k is the entire stable category, even though the thick subcategory generated by k (without closure under arbitrary tensors) is not.

The context here is the stable category $\mathsf{stmod}(kG)$ of finitely generated kG-modules. This is a tensor triangulated category with the tensor being $\otimes = \otimes_k$, the action of the group on a tensor product of modules being given by the usual diagonal Hopf algebra structure $g \mapsto g \otimes g$ for $g \in G$. It is the quotient category of the ordinary category of kG-modules and kG-homomorphisms by the projective modules. That is, the maps of kG-modules which are zero in $\mathsf{stmod}(kG)$ are those that factor through some projective module. We call these *null* maps. A map $f\colon N \to M$ is said to be tensor nilpotent if for some value n, $f^{\otimes n}\colon N^{\otimes n} \to M^{\otimes n}$ is null.

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As with many notions in modular representation theory, tensor nilpotence is controlled by elementary abelian p-subgroups. Using some rather well known methods, we prove in Section 2 that there is a constant C, depending only on G, such that if $f\colon N \to M$ is a map of kG-modules which is null on restriction to every elementary abelian p-subgroup E of G then $f^{\otimes C}\colon N^{\otimes C} \to M^{\otimes C}$ is null. On the other hand, even for an elementary abelian p-group E there is no bound on the possible tensor nilpotence of maps in the stable module category. We give examples for the Klein four group $\mathbb{Z}/2 \times \mathbb{Z}/2$ of maps with arbitrarily large degree of tensor nilpotence.

The same group provides examples of maps $f\colon k \to M$ where the question of whether f is tensor nilpotent depends on the Hopf algebra structure we put on kE. In our example, M is two dimensional. With the group theoretic Hopf algebra structure $f^{\otimes 3}$ is null, whereas with the Hopf algebra structure coming from regarding kE as a restricted enveloping of a restricted p-Lie algebra, no power of f is null.

We develop a related notion of *strong nilpotence* of maps from the trivial module over elementary abelian groups. This concept concerns factorization through the second socle of the projective module and is better behaved than ordinary nilpotence. In Section 6 we prove the following.

Theorem 1.1. (i) *If a map of kG-modules is strongly nilpotent then it is tensor nilpotent.*

(ii) *The strong nilpotence of a particular map $f\colon k \to M$ (or $f\colon N \to k$) does not depend on the Hopf algebra structure on kG.*

(iii) *There is a bound C depending only on G and not on f such that if f is strongly nilpotent, then $f^{\otimes C}$ is null.*

In the latter part of the paper, we apply these notions to questions about ghosts and generation. A *tensor M-ghost* is a map $f\colon X \to Y$ such that for all modules U,

$$f_*\colon \widehat{\mathrm{Ext}}^*_{kG}(M \otimes U, X) \to \widehat{\mathrm{Ext}}^*_{kG}(M \otimes U, Y)$$

is the zero map. It turns out that this is equivalent to the statement that

$$\mathrm{Id}_M \otimes f\colon M \otimes X \to M \otimes Y$$

is null. For example, a tensor k-ghost is the same as a null map. Given a kG-module M, take the adjoint $k \to M \otimes M^*$ of the identity map on M, and complete to a triangle

$$N \xrightarrow{f} k \to M \otimes M^*$$

in $\mathsf{stmod}(kG)$. Then f is a tensor M-ghost. The *tensor M-ghost number* is the smallest n such that the composite of n tensor M-ghosts is always null, or infinity if there is no such n.

For a kG-module M, the *tensor M-generation number* is the smallest n such that every module can be built in at most n steps from modules of the form $M \otimes X$, or infinity if there is no such n. These notions are related by the following theorem, proved in Section 8.

Theorem 1.2. *Given a kG-module M, form the triangle $N \xrightarrow{f} k \to M \otimes M^*$ in $\mathsf{stmod}(kG)$ as above. Then the following statements are equivalent.*

(i) *The tensor M-generation number is at most n.*

(ii) *The map $f^{\otimes n}$ is null.*

(iii) *The tensor M-ghost number is at most n.*

These statements hold for some $n \geq 1$ if and only if the support variety $V_G(M)$ is equal to $V_G(k)$.

This leads us to the following question, which we study in Section 9.

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