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Enhancing the filtered derived category

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ABSTRACT

The filtered derived category of an abelian category has played a useful role in subjects including geometric representation theory, mixed Hodge modules, and the theory of motives. We develop a natural generalization using current methods of homotopical algebra, in the formalisms of stable ∞-categories, stable model categories, and pretriangulated, idempotent-complete dg categories. We characterize the filtered stable ∞-category $Fil(\mathcal{C})$ of a stable ∞-category $\mathcal{C}$ as the left exact localization of sequences in $\mathcal{C}$ along the ∞-categorical version of completion (and prove analogous model and dg category statements). We also spell out how these constructions interact with spectral sequences and monoidal structures. As examples of this machinery, we construct a stable model category of filtered $\mathcal{D}$-modules and develop the rudiments of a theory of filtered operads and filtered algebras over operads. This paper is also available at arXiv:1602.01515v3.

1. Introduction

The filtered derived category plays an important role in the setting of constructible sheaves, $\mathcal{D}$-modules, mixed Hodge modules, and elsewhere. Our goal is to revisit this construction using current technology for homotopical algebra, extending it beyond the usual setting of chain complexes (or sheaves of chain complexes). Put simply, we would like to describe the filtered version of a stable ∞-category, generalizing this classical situation. We also develop explicitly this machinery in the setting of model categories and dg categories, so that it can be deployed in highly structured contexts and used in concrete computations.

There are several places where this kind of machinery would be useful. For instance, it applies to filtered spectra (and sheaves of spectra). In a different direction, this work allows one to work with filtered ∞-operads and filtered algebras over ∞-operads in a clean way, as explained in §6.

Several results here are undoubtedly well-known but seem to be unavailable for convenient reference. In this introduction, we begin by describing the classical construction and the basic problem we pursue.

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We then describe our main results and the structure of the paper and finish with a comparison to other work.

1.1. The classical construction

Let $\mathbf{Z}$ denote the integers, equipped with the usual total ordering by $<$. Let $\underline{\mathbf{Z}}$ denote the associated category, whose objects are integers and where $\underline{\mathbf{Z}}(m, n)$ is empty if $m > n$ and is a single element if $m \leq n$.

Let $\mathtt{A}$ denote an abelian category.

Definition 1.2. The category of *sequences in* $\mathtt{A}$ is the functor category $\mathrm{Fun}(\underline{\mathbf{Z}}, \mathtt{A})$. We denote it by $\mathrm{Seq}(\mathtt{A})$.

Definition 1.3. The *filtered category of* $\mathtt{A}$, denoted $\mathrm{Fil}(\mathtt{A})$, is the full subcategory of the category of sequences in which an object $X \colon \underline{\mathbf{Z}} \to \mathtt{A}$ satisfies the condition that $X(m \to n) \colon X(m) \to X(n)$ is a monomorphism for every $m \leq n$.

Given an object X in $\mathrm{Fil}(\mathtt{A})$, we view it as equipping the object $X(\infty) = \operatorname{colim} X$ with the filtration whose nth component is $X(n)$. Thus, we only consider filtrations that are "exhaustive" in the classical terminology.

Remark 1.4. Often, people are interested in a *tower*, i.e., a sequence where one is interested in the limit $X(-\infty)$ (or homotopy limit) rather than colimit. In algebra, one often studies examples where each structure map is an epimorphism, such as

$$\cdots \to \mathbf{Z}/p^n\mathbf{Z} \to \mathbf{Z}/p^{n-1}\mathbf{Z} \to \cdots \to \mathbf{Z}/p^2\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0.$$

In stable homotopy theory, one studies towers of spectra, such as the chromatic tower. It is possible to view a tower as a sequence in our sense by working in the opposite category (see Remark 2.20).

It is well-known that $\mathrm{Fil}(\mathtt{A})$ is additive but not abelian, which increases the complexity of homological algebra in this setting.

Definition 1.5. The *associated graded* functor $\mathrm{Gr} \colon \mathrm{Seq}(\mathtt{A}) \to \prod_{\mathbf{Z}} \mathtt{A}$ sends a sequence X to $\mathrm{Gr}\, X$ where

$$(\mathrm{Gr}\, X)_n = X(n)/ \operatorname{im} X(n-1 \to n)$$

for all $n \in \mathbf{Z}$.

Remark 1.6. In Lemma 3.30, we show that Gr is left adjoint to the functor that turns a list of objects $(A_n)_{n\in\mathbf{Z}}$ into a sequence where every structure map is zero.

We now consider the abelian category $\mathrm{Ch}(\mathtt{A})$ of unbounded chain complexes in $\mathtt{A}$. We equip it with the quasi-isomorphisms as its class of weak equivalences.

Definition 1.7. The *filtered derived category of* $\mathtt{A}$, denoted $\mathrm{D}^{\mathrm{fil}}(\mathtt{A})$, is the localization of $\mathrm{Ch}(\mathrm{Fil}(\mathtt{A}))$ (equivalently, $\mathrm{Fil}(\mathrm{Ch}(\mathtt{A}))$) with respect to the *filtered weak equivalences*, which are maps of sequences $f \colon X \to Y$ such that $\mathrm{Gr}\, f \colon \mathrm{Gr}\, X \to \mathrm{Gr}\, Y$ is an indexwise quasi-isomorphism.

A core objective of this paper is to formulate and analyze a version of this construction that takes as input a stable ∞-category or stable model category. We show, of course, that our construction applied to