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Gaussian elements of a semicontent algebra

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A R T I C L E I N F O

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A B S T R A C T

The connection between a univariate polynomial having locally principal content and the content function acting like a homomorphism (the so-called Gaussian property) has been explored by many authors. In this work, we extend several such results to the contexts of multivariate polynomials, power series over a Noetherian ring, and base change of affine K-algebras by separable algebraically closed field extensions. We do so by using the framework of the Ohm–Rush content function. The correspondence is particularly strong in cases where the base ring is approximately Gorenstein or the element of the target ring is regular.

1. Introduction

Gauss's lemma is fundamental in number theory and algebra. If R is a commutative ring and $f \in R[x]$, the *content* $\mathrm{c}(f)$ of f is the ideal in R generated by the R-coefficients of f. Gauss showed that when $R = \mathbb{Z}$, we always have

$$\mathrm{c}(f)\,\mathrm{c}(g) = \mathrm{c}(fg). \tag{1}$$

Along with various avatars, it may be used to explore unique factorization and primitivity of polynomials. However, in some sense it almost never holds. In fact [19], for an integral domain R, (1) holds for *all* pairs of polynomials $f, g \in R[X]$ if and only if R is a Prüfer domain (a condition trivially satisfied by $\mathbb{Z}$). From this perspective, it makes sense to ask for properties of polynomials that satisfy Gauss's lemma. One says that $f \in R[X]$ is *Gaussian* if for all $g \in R[X]$, (1) holds. It turns out that even this condition is close to the Prüfer condition. For a reduced ring R [11], and also for an approximately Gorenstein Noetherian ring R [7], $f \in R[X]$ is Gaussian if and only if $c(f)$ is *locally principal*. Moreover, for any commutative ring R [10], a regular element $f \in R[X]$ is Gaussian if and only if $c(f)$ is locally principal.

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In [17,18,14,15], conditions were established and developed on a ring extension such that the notion of the content of an element of the target ring S as an ideal in the base ring R is a useful construct. As such, the Gaussian property of an element of S with respect to R makes sense, and one would hope to come to similar conclusions in this expanded context. Examples of "semicontent algebras" (defined in [4] as a generalization of "content algebras" [17]) include

- affine semigroup extensions [16], including polynomial extensions in several variables,
- power series extensions of Noetherian rings [3], and
- base change of affine K-algebras (K a field) by very well-behaved field extensions L/K [5, Propositions 3.8 and 3.11].

In the first two cases above, the content of an element is the ideal generated by the coefficients of the polynomial-analogue of the extension with respect to the base ring. In the third case, there's a more interesting answer based on the identity of the target ring as a free module over the base ring. See the discussion following Proposition 5.4.

The core results of this paper (see Section 3) generalize the results of [7] to the more general context of the Ohm–Rush content function. However the casual reader who is less well versed in Ohm–Rush content theory may find the corollaries to these theorems (see Section 5), which are stated in a more specific context, to be of primary interest. In particular, the following is an incomplete representation of what we have proved, being a proper subset of the corollaries of our core results (see Theorems 5.1–5.3 and Corollaries 5.6–5.7):

Main Results. *Let R be a commutative ring, and let S be an R-algebra and $f \in S$ as below. Then under any of the following conditions, $\mathrm{c}(f)$ is locally principal if and only if Equation (1) is satisfied for every $g \in S$:*

I. *R is Noetherian and approximately Gorenstein, and $S = R[x_1, \ldots, x_n]$ or any other affine semigroup algebra over R.*
II. *R is locally Noetherian, $S = R[x_1, \ldots, x_n]$ or any other affine semigroup algebra over R, and f is regular.*
III. *R is Noetherian and approximately Gorenstein, and $S = R[\![x_1, \ldots, x_n]\!]$.*
IV. *R is a finitely generated artinian Gorenstein K-algebra, where K is a field, $L = K(y_1, \ldots, y_t)$ is a purely transcendental field extension, and $S = R \otimes_K L$, where "content" is with respect to the field variables y_j.*
V. *R is a finitely generated K-algebra, where K is an algebraically closed field, L/K is any field extension, f is regular, and $S = R \otimes_K L$, where "content" is with respect to a vector space basis of L over K.*

The paper is structured as follows: In §2, we recall the framework built up so far about the Ohm–Rush content function. This allows us a language in which to develop our core results.

In §3, we build Ohm–Rush theory up a bit more, in service of the core theorems of the paper. What we investigate in this section is the connection between locally principal content and Gaussianness. One direction (Theorem 3.1) is relatively easy. This first core theorem says that in semicontent algebras, an element of the algebra that has locally principal content will always be Gaussian. The second of the core theorems (Theorem 3.5) gives a partial converse. It says that if R is approximately Gorenstein, and S is semicontent over R, then any Gaussian element of S has locally principal content in R. Then in the third core theorem, Theorem 3.11, we weaken the hypothesis on R (just locally Noetherian) but strengthen the hypotheses on the R-algebra S (requiring it to be free as an R-module). We show that in this case, any *regular* Gaussian element of S has locally principal content ideal.

In §4, we obtain in Theorem 4.2 a new characterization of Prüfer domains based on the Gaussian property. Finally in §5, we apply the built up theorems to draw connections between Gaussian elements and locally