ARTICLE IN PRESS JPAA:5799

Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Embeddings of rank-2 tori in algebraic groups

Neha Hooda

Indian Statistical Institute, 7-S.J.S. Sansanwal Marg, New Delhi 110016, India

ARTICLE INFO

Article history:
Received 15 April 2017
Received in revised form 7 October 2017
Available online xxxx
Communicated by S. Koenig

ABSTRACT

Let k be a field of characteristic different from 2 and 3. In this paper we study connected simple algebraic groups of type A_2, G_2 and F_4 defined over k, via their rank-2 k-tori. Simple, simply connected groups of type A_2 play a pivotal role in the study of exceptional groups and this aspect is brought out by the results in this paper. We refer to tori, which are maximal tori of A_n type groups, as unitary tori. We discuss conditions necessary for a rank-2 unitary k-torus to embed in simple k-groups of type A_2, G_2 and F_4 in terms of the mod-2 Galois cohomological invariants attached with these groups. The results in this paper and our earlier paper ([6]) show that the mod-2 invariants of groups of type G_2, F_4 and A_2 are controlled by their k-subgroups of type A_1 and A_2 as well as the unitary k-tori embedded in them.

1. Introduction

The main aim of the paper is to study exceptional algebraic groups via their subgroups. This theme has been widely explored by various authors (Martin Liebeck, Gary Seitz, Adam Thomas, Donna Testerman to mention a few), mainly for **split** groups ([15], [16], [12], [27]). When the field of definition k of the concerned algebraic groups is not algebraically closed, the classification of k-subgroups is largely an open problem.

Let K be an algebraically closed field. The classification of semisimple algebraic groups over K is well understood.

Theorem 1.1 *(Chevalley classification theorem). Two semisimple linear algebraic groups are isomorphic if and only if they have isomorphic root data. For each root datum there exists a semisimple algebraic group which realizes it.*

The simple algebraic groups have irreducible root systems or equivalently, have connected Dynkin diagrams. Irreducible root systems fall into nine types, called the **Cartan–Killing types**, labeled as $A_n, B_n, C_n, D_n, E_6, E_7, F_4, G_2$. The first four types exist for each natural number n (with the exception

E-mail address: neha.hooda2@gmail.com.

https://doi.org/10.1016/j.jpaa.2017.11.013
0022-4049/

Please cite this article in press as: N. Hooda, Embeddings of rank-2 tori in algebraic groups, J. Pure Appl. Algebra (2017), https://doi.org/10.1016/j.jpaa.2017.11.013

of B_1, C_1, D_1 and D_2), while the remaining five types are just one in each case. Simple groups with root system or Dynkin diagram of types A_n, B_n, C_n, D_n are called **classical groups** and the simple groups with root systems of type E_6, E_7, E_8, F_4, G_2 are called **exceptional groups**. Let G be a simple algebraic group over a field k. By the type of G we mean the Cartan–Killing type of the root system of the group $G \otimes \overline{k}$, obtained by extending scalars to an algebraic closure $\overline{k}$ of k.

Let G be a simple linear algebraic group over K. Then corresponding to any subdiagram of the Dynkin diagram of G, there exists a subgroup of G which realizes it, i.e. has the subdiagram as its Dynkin diagram. But this fails to hold for a non-algebraically closed field. For example, over a non-algebraically closed field k a connected simple algebraic group G may not have any subgroup of type A_1, though the Dynkin diagram of G always has A_1 as a subdiagram (see Remark 4.2). Hence over a non-algebraically closed field k, it is important to know what are all simple k-subgroups of G.

In our earlier paper ([6]) we answer this for groups of type A_2, G_2 and F_4. **We prove that when G is a k-group of type F_4 (resp. G_2) arising from an Albert (resp. octonion) division algebra then the possible type of a simple k-subgroup of G is A_2 or D_4 (resp. A_1 or A_2).** We further studied k-embeddings of connected, simple algebraic groups of type A_1 and A_2 in simple groups of type G_2 and F_4, defined over k, in terms of their respective mod-2 Galois cohomological invariants. We showed that these groups are generated by their A_2 type (and A_1 type) k-subgroups. Owing to these results, importance of groups of type A_1, A_2 becomes evident in studying exceptional groups. The theme of irreducible subgroups and A_1-type subgroups of algebraic groups has been thoroughly investigated by several authors over algebraically closed fields and finite fields, see for example ([14], [24], [25], [26], [11], [12], [28], [13]). Since the Galois cohomological invariants of any group of type G_2 and F_4 over finite fields or algebraically closed fields are all trivial, our results are valid also over such fields. Thus the knowledge of simple k-subgroups is a useful tool in studying these groups. This motivated us for our research.

The main aim of this paper is to investigate embeddings of rank-2 tori in groups of type G_2, F_4 and A_2 via the mod-2 invariants of these groups. To a simple, simply connected algebraic group G of type F_4, G_2 or A_2 defined over k, one attaches certain mod-2 Galois cohomological invariants, which are the Arason invariants of some Pfister forms attached to these groups. Let G be a group of type F_4 defined over k. Then there exists an Albert algebra A over k such that $G = \mathbf{Aut}(A)$, the full group of automorphisms of A. To any Albert algebra A, one attaches a certain reduced Albert algebra $\mathcal{H}_3(C, \Gamma)$, for an octonion algebra C over k and $\Gamma = Diag(\gamma_1, \gamma_2, \gamma_3) \in GL_3(k)$ ([19]). This defines two mod-2 invariants for $G = \mathbf{Aut}(A)$:

$$f_3(G) = f_3(A) := e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z}),$$
$$f_5(G) = f_5(A) := e_5(n_C \otimes << -\gamma_1^{-1}\gamma_2, -\gamma_2^{-1}\gamma_3 >>) \in H^5(k, \mathbb{Z}/2\mathbb{Z}),$$

where e_3 and e_5 are the Arason invariants of the respective Pfister forms and n_C is the norm form of C. We set $Oct(G) = Oct(A) := C$. Similarly, an algebraic group of type G_2 defined over k is precisely of the form $\mathbf{Aut}(C)$, for an octonion algebra C over k with norm form n_C, and this is classified by the Arason invariant $f_3(G) = e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$. Define $Oct(G) := C$. Finally, let G be a simple, simply connected group of type A_2 defined over k. To such a group G, one attaches an invariant $f_3(G) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$, the Arason invariant of a 3-fold Pfister form over k, which is the norm form of an octonion algebra C. We define $Oct(G) := C$ (see §2.5).

Let L, K be étale algebras over k of dimensions $3, 2$ resp. and $T = \mathbf{SU}(L \otimes K, 1 \otimes ^-)$, where $^-$ denotes the non-trivial involution on K. Then T is a torus defined over k, referred to in the paper as the K-**unitary torus** associated with the pair (L, K). For this torus, we let $q_T := < 1, -\alpha\delta > = N_{k(\sqrt{\alpha\delta})/k}$, where $Disc(L) = k(\sqrt{\delta})$ and $K = k(\sqrt{\alpha})$. Such tori are important as they occur as maximal tori in simple, simply connected groups of type A_2 and G_2 (cf. [29], [30], [2]). We will be interested in conditions under which such tori embed in groups of type A_2, G_2 or F_4 defined over k. A unitary torus T will be called **distinguished** over k if q_T is hyperbolic over k, or equivalently, if $Disc(L) = K$. We shall see that the behavior of the invariant f_3 for

Please cite this article in press as: N. Hooda, Embeddings of rank-2 tori in algebraic groups, J. Pure Appl. Algebra (2017), https://doi.org/10.1016/j.jpaa.2017.11.013