ARTICLE IN PRESS JPAA:5800

Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

On the rigidity of moduli of weighted pointed stable curves

Barbara Fantechi [a], Alex Massarenti [b,*]

[a] SISSA, via Bonomea 265, 34136 Trieste, Italy
[b] Universidade Federal Fluminense, Campus Gragoatá, rua Alexandre Moura 8 – São Domingos, 24210-200, Niterói, Rio de Janeiro, Brazil

ARTICLE INFO

Article history:
Received 20 May 2017
Received in revised form 3 October 2017
Available online xxxx
Communicated by S. Kovács

MSC:
Primary: 14H10; secondary: 14D22; 14D23; 14D06

ABSTRACT

Let $\overline{\mathcal{M}}_{g,A[n]}$ be the Hassett moduli stack of weighted stable curves, and let $\overline{M}_{g,A[n]}$ be its coarse moduli space. These are compactifications of $\mathcal{M}_{g,n}$ and $M_{g,n}$ respectively, obtained by assigning rational weights $A = (a_1, ..., a_n)$, $0 < a_i \leq 1$ to the markings; they are defined over $\mathbb{Z}$, and therefore over any field. We study the first order infinitesimal deformations of $\overline{\mathcal{M}}_{g,A[n]}$ and $\overline{M}_{g,A[n]}$. In particular, we show that $\overline{M}_{0,A[n]}$ is rigid over any field, if $g \geq 1$ then $\overline{\mathcal{M}}_{g,A[n]}$ is rigid over any field of characteristic zero, and if $g + n > 4$ then the coarse moduli space $\overline{M}_{g,A[n]}$ is rigid over an algebraically closed field of characteristic zero. Finally, we take into account a degeneration of Hassett spaces parametrizing rational curves obtained by allowing the weights to have sum equal to two. In particular, we consider such a Hassett 3-fold which is isomorphic to the Segre cubic hypersurface in $\mathbb{P}^4$, and we prove that its family of first order infinitesimal deformations is non-singular of dimension ten, and the general deformation is smooth.

Introduction

In [9] B. Hassett introduced new compactifications $\overline{\mathcal{M}}_{g,A[n]}$ of the moduli stack $\mathcal{M}_{g,n}$ parametrizing smooth genus g curves with n marked points, where the notion of stability is defined in terms of a fixed vector of rational weights $A[n] = (a_1, ..., a_n)$, on the markings. The classical Deligne–Mumford compactification corresponds to the weights $a_1 = ... = a_n = 1$; Hassett construction requires that $0 < a_i \leq 1$ for every i and that $\sum a_i > 2 - 2g$.

As the stack $\overline{\mathcal{M}}_{g,n}$, the stacks $\overline{\mathcal{M}}_{g,A[n]}$ are smooth and proper over $\mathbb{Z}$, and therefore $\overline{\mathcal{M}}^R_{g,A[n]}$ is defined over any commutative ring R via base change. By [13] the formation of the coarse moduli space is compatible with flat base change; we write $\overline{M}^R_{g,n}$ for the coarse moduli scheme of $\overline{\mathcal{M}}^R_{g,n}$, and refer to it as a Hassett moduli space. Again in analogy with the Deligne–Mumford case, Hassett stacks for $g = 0$ are already schemes, hence coincide with the corresponding Hassett spaces.

* Corresponding author.
E-mail addresses: fantechi@sissa.it (B. Fantechi), alexmassarenti@id.uff.br (A. Massarenti).

https://doi.org/10.1016/j.jpaa.2017.11.014
0022-4049/© 2017 Elsevier B.V. All rights reserved.

Please cite this article in press as: B. Fantechi, A. Massarenti, On the rigidity of moduli of weighted pointed stable curves, J. Pure Appl. Algebra (2017), https://doi.org/10.1016/j.jpaa.2017.11.014

Hassett spaces are central objects in the study of the birational geometry of $\overline{M}_{g,n}$. Indeed, in genus zero some of these spaces appear as intermediate steps of the blow-up construction of $\overline{M}_{0,n}$ developed by M. Kapranov in [11] and some of them turn out to be Mori Dream Spaces [1, Section 6], while in higher genus they may be related to the LMMP on $\overline{M}_{g,n}$ [20].

In this paper we push forward the techniques developed in [6] to study the infinitesimal deformations of Hassett moduli stacks and spaces over an arbitrary field. The results in Theorems 2.5 and 2.7 can be summarized as follows.

Theorem 1. *Let K be an arbitrary field, and $n \geq 3$ an integer. Then the genus zero Hassett moduli space $\overline{M}^K_{0,A[n]}$ is rigid for any vector of weights $A[n]$.*

Let $g \geq 1$ and assume K is a field of characteristic zero. Then Hassett stack $\overline{\mathcal{M}}^K_{g,A[n]}$ is rigid for any vector of weights $A[n]$.

For a field K of characteristic zero we then apply the deformation theory of varieties with transversal A_1 and $\frac{1}{3}(1,1)$ singularities developed in [6, Sections 5.4, 5.5] to the study of infinitesimal deformations of the coarse moduli spaces $\overline{M}^K_{g,A[n]}$. In the following statement we summarize the result on deformations of $\overline{M}^K_{g,A[n]}$ in Proposition 2.6 and Theorem 2.7.

Theorem 2. *Let K be a field of characteristic zero. If $g+n \geq 4$ then the coarse moduli space $\overline{M}^K_{g,A[n]}$ does not have non-trivial locally trivial first order infinitesimal deformations for any vector of weights $A[n]$.*

If K is an algebraically closed field of characteristic zero and $g+n > 4$ then $\overline{M}^K_{g,A[n]}$ is rigid for any vector of weights $A[n]$.

In Section 3 we consider a natural variation $\overline{M}_{0,\widetilde{A}[n]}$ on the moduli problem of weighted pointed rational curves, introduced by B. Hassett in [9, Section 2.1.2] by allowing the weights to have sum equal to two.

In particular, we consider Hassett space $\overline{M}_{0,\widetilde{A}[6]}$ with weights $a_1 = ... = a_6 = \frac{1}{3}$. This space is isomorphic to the Segre cubic, a 3-fold of degree three in $\mathbb{P}^4$ with ten nodes which carries a very rich projective geometry [3]. In Section 3 we study the infinitesimal deformations of $\overline{M}_{0,\widetilde{A}[6]}$, that is of the Segre cubic.

In Theorem 3.2 we prove that $\overline{M}_{0,\widetilde{A}[6]}$ does not have non-trivial locally trivial deformations, while its family of first order infinitesimal deformations is non-singular of dimension ten and the general deformation is smooth.

Finally, in Section 2.1 we apply the rigidity results in Section 2, and the techniques developed in [6, Section 1] to lift automorphisms from zero to positive characteristic, in order to extend the main results on the automorphism groups of Hassett spaces in [18], [19], [2], [16] and [17] over an arbitrary field.

We would like to stress that A. Duncan and Z. Reichstein used the mentioned results on the automorphism group of $\overline{M}_{g,n}$ to study the unirationality of its forms on a field of characteristic zero [4, Section 6]. We believe that similarly our results in Section 2.1 may be applied to the study of forms of moduli spaces of weighted pointed stable curves.

Acknowledgments. The authors are members of the Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni of the Istituto Nazionale di Alta Matematica “F. Severi” (GNSAGA-INDAM). We thank the referee for his comments that helped us to improve a first version of the paper.

1. Preliminaries on Hassett moduli spaces

Let S be a noetherian scheme and g, n two non-negative integers. A family of nodal curves of genus g with n marked points over S consists of a flat proper morphism $\pi : C \to S$ whose geometric fibers

Please cite this article in press as: B. Fantechi, A. Massarenti, On the rigidity of moduli of weighted pointed stable curves, J. Pure Appl. Algebra (2017), https://doi.org/10.1016/j.jpaa.2017.11.014