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Exact completion of path categories and algebraic set theory Part I: Exact completion of path categories

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ABSTRACT

We introduce the notion of a "category with path objects", as a slight strengthening of Kenneth Brown's classical notion of a "category of fibrant objects". We develop the basic properties of such a category and its associated homotopy category. Subsequently, we show how the exact completion of this homotopy category can be obtained as the homotopy category associated to a larger category with path objects, obtained by freely adjoining certain homotopy quotients. In a second part of this paper, we will present an application to models of constructive set theory. Although our work is partly motivated by recent developments in homotopy type theory, this paper is written purely in the language of homotopy theory and category theory, and we do not presuppose any familiarity with type theory on the side of the reader.

1. Introduction

The phrase "path category" in the title is short for "category with path objects" and refers to a modification of Kenneth Brown's notion of a category of fibrant objects [8], originally meant to axiomatise the homotopical properties of the category of simplicial sheaves on a topological space. Like categories of fibrant objects, path categories are categories equipped with classes of fibrations and weak equivalences, and as such they are closely related to Quillen's model categories which have an additional class of cofibrations [25,26,19]. Our modification of Brown's definition mainly consists in an additional axiom which in the language of Quillen model categories would amount to the condition that every object is cofibrant. One justification for this modification is that there still are plenty of examples. One source of examples is provided by taking the fibrant objects in a model category in which all objects are cofibrant, such as the category of simplicial sets, or the categories of simplicial sheaves equipped with the injective model structure. More generally, many

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model categories have the property that objects over a cofibrant object are automatically cofibrant. For example, this holds for familiar model category structures for simplicial sets with the action of a fixed group, for dendroidal sets, and for many more. In such a model category, the fibrations and weak equivalences between objects which are both fibrant and cofibrant satisfy our modification of Brown's axioms.

Another justification, and in fact our main motivation, for this modification of Brown's axioms is that these modified axioms are satisfied by the syntactic category constructed out of a type theory [2,15]. Thus, our work builds on the recently discovered interpretation of Martin–Löf's theory of identity types Quillen model categories [3]. This interpretation has been extended by Voevodsky to an interpretation of Martin–Löf type theory in the category of simplicial sets [22] (see also [7,14,16,28,29]).

In addition, our work is relevant for constructive set theory. Aczel has provided an interpretation of the language of set theory in a type theory with a suitable universe [1], and the question arises whether it is possible to construct models of set theory out of certain path categories. We will turn to this question in Part II of this paper.

The precise contents of this paper are as follows. In Section 2 we introduce the notion of a path category and verify that many familiar constructions from homotopy theory can be performed in such path categories and retain their expected properties. It is necessary for what follows to perform this verification, but there is very little originality in it. An exception is perhaps formed by our construction of suitable path objects carrying a connection structure as in Theorem 2.28 and our statement concerning the existence of diagonal fillers which are half strict, half up-to-homotopy, as in Theorem 2.38 below. We single out these two properties here also because they play an important rôle in later parts of the paper.

In Section 3 we will introduce a notion of "homotopy exact completion" for such path categories, a new category obtained by freely adjoining certain homotopy quotients. For "trivial" path categories in which every map is a fibration and only isomorphisms are weak equivalences this notion of homotopy exact completion coincides with the ordinary notion of exact completion, well known from category theory (see [9,10,13]). In case the path category is obtained from the syntax of type theory this coincides with what is known as the setoids construction (see [4]). Indeed, the type-theorist can think of our work as a categorical analysis of this construction informed by the homotopy-theoretic interpretation of type theory. The main result in Section 3 shows that the exact completion of the homotopy category of a path category $\mathcal{C}$ is itself a homotopy category of another path category which we call $\mathrm{Ex}(\mathcal{C})$, see Proposition 3.18 and Theorem 3.14 below.

In Section 4 we show that if $\mathcal{C}$ has homotopy sums which are, in a suitable sense, stable and disjoint, then the homotopy exact completion is a pretopos (see Theorem 4.10). We will also show that the homotopy exact completion has a natural numbers object if $\mathcal{C}$ has what we will call a homotopy natural numbers object.

Finally, in Section 5 we will show that the homotopy exact completion improves the properties of the original category in that it will satisfy certain extensionality principles even when the original category does not. This is analogous to what happens for ordinary exact completions: the ordinary exact completion $\mathcal{C}'$ of a category $\mathcal{C}$ will be locally Cartesian closed (that is, will have internal homs in every slice) whenever $\mathcal{C}$ has this property in a weak form, where weak is meant to indicate that one weakens the usual universal property of the internal hom by dropping the uniqueness requirement, only keeping existence (see [12]). In the same vein we show in Section 5 that if a path category has weak homotopy Π-types (i.e. weak fibrewise up-to-homotopy internal homs) then its exact completion has exponentials in every slice. In type-theoretic terms this means that the homotopy exact completion will always satisfy a form of function extensionality; something similar holds for the path category $\mathrm{Ex}(\mathcal{C})$.

At this point it is probably good to add a few words about our approach and how it relates to some of the work that is currently being done at the interface of type theory and homotopy theory. First of all, we take a resolutely categorical approach; in particular, no knowledge of the syntax of type theory is required to understand this paper. As a result, we expect our paper to be readable by homotopy theorists.