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Free skew monoidal categories

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ABSTRACT

In the paper *Triangulations, orientals, and skew monoidal categories*, the free skew monoidal category **Fsk** on a single generating object was described. We sharpen this by giving a completely explicit description of **Fsk**, and so of the free skew monoidal category on any category. As an application we describe adjunctions between the operad for skew monoidal categories and various simpler operads. For a particular such operad $\mathcal{L}$, we identify skew monoidal categories with certain colax $\mathcal{L}$-algebras.

1. Introduction

A skew monoidal category is a category $\mathcal{C}$ equipped with a functor $\mathcal{C}^2 \to \mathcal{C}$ whose effect on objects we write as $(a, b) \mapsto ab$, an object $i \in \mathcal{C}$, and natural transformations

$$(ab)c \xrightarrow{\alpha} a(bc)$$
$$ia \xrightarrow{\lambda} a$$
$$a \xrightarrow{\rho} ai$$

satisfying five coherence conditions. When the maps α, ρ, and λ are invertible, we recover the usual notion of monoidal category.

While this might seem like a mindless generalisation, it turns out that there are important examples of skew monoidal categories which are not monoidal. The first such class of examples arises from quantum algebra, and is due to Szlachányi [9]: he realised that bialgebroids can be described using skew monoidal categories. Specifically, a bialgebroid with base ring R (not necessarily commutative) is the same thing as a skew monoidal closed structure on the category R-**Mod** of R-modules.

A second class of examples arises from the intersection of homotopical algebra and 2-category theory: a host of naturally occurring skew monoidal closed structures on Quillen model categories that arise in

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2-dimensional universal algebra were described in [1]. These examples are monoidal in a homotopical sense, in that they yield genuine monoidal closed structures on the associated homotopy categories.

Unlike the situation for monoidal categories, it is not the case for skew monoidal categories that all diagrams built up out of the structure maps commute: for example, the composite

$$ii \xrightarrow{\lambda} i \xrightarrow{\rho} ii$$

is not the identity, and so the "coherence problem" for skew monoidal categories is not a trivial one. One way to formulate this coherence problem is to ask what is the *free skew monoidal category* on a given category. An answer to this question was given in [8].

As was observed in [8], the structure of a skew monoidal category is *clubbable*, in the sense of [6]; equivalently, it can be given in terms of a *plain operad in* **Cat**, where by "plain", we mean that there are no actions of the symmetric groups. It then follows that in order to describe the free skew monoidal category on a general category $\mathcal{C}$ it suffices to do it on the terminal category, and in fact this is what is done in [8].

The free skew monoidal category on an object, called **Fsk** in [8], is determined by the following universal property. There is a designated object $X \in$ **Fsk** ("the generator") and for any skew monoidal category $\mathcal{C}$, evaluation at X determines a bijection between the set of (strict) monoidal functors from **Fsk** to $\mathcal{C}$ and objects of $\mathcal{C}$.

An example of a skew monoidal category is the category $\mathbf{Ord}_\perp$ of finite non-empty ordinals, with morphisms the functions which preserve both order and bottom element. The product is given by ordinal sum, and the unit object is the ordinal $\mathbf{1} = \{0\}$. This is strictly associative, but the maps λ and ρ are non-invertible. By its universal property, the free skew monoidal category **Fsk** on one object has a unique structure-preserving functor to $\mathbf{Ord}_\perp$ which sends the generator to $\mathbf{1}$. A key result of [8] was that this functor is faithful, so that the morphisms of **Fsk** can be represented as certain functions between finite sets.

While the objects of **Fsk** were described in an entirely explicit way, the morphisms were not. The main goal of this paper is to remedy this, by giving a completely explicit condition characterising the morphisms.

As an application, we construct various adjunctions between the operad $\mathcal{S}$ for skew monoidal categories and various simpler operads $\mathcal{T}$. In each case we have an operad map $F\colon \mathcal{S} \to \mathcal{T}$ and we show that F has a left or right adjoint in each component. By the usual "doctrinal adjunction" results [5] this enables us to view skew monoidal categories as colax/lax $\mathcal{T}$-algebras. When $\mathcal{T}$ is the terminal operad, the unique map $F\colon \mathcal{S} \to \mathcal{T}$ has both adjoints, and so any skew monoidal category yields both a colax monoidal category and a lax monoidal category. These processes lose structure, but choosing for $\mathcal{T}$ an only slighter more complex operad $\mathcal{L}$, we find that colax $\mathcal{L}$-algebras encode the skew monoidal structure entirely. These results are used in our companion paper [2] which introduces and studies *skew multicategories*, the multicategorical analogue of skew monoidal categories.

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2. Background on clubs and operads

In this section we group together various facts about plain **Cat**-operads and clubs over $\mathbb{N}$. There is nothing particularly original here, but we could not find any convenient reference containing everything we need, which largely amounts to combining aspects of [6], [7], and [4].

Let $\mathbb{N}$ denote the discrete category with objects the natural numbers. The functor 2-category $[\mathbb{N}, \mathbf{Cat}]$ has a monoidal structure with tensor product $\circ$ given by $(A \circ B)_n = \sum_{n=n_1+\ldots+n_k} A_k \times B_{n_1} \times \ldots \times B_{n_k}$ and with unit J given by $J_1 = 1$ and $J_n = 0$ if $n \neq 1$.