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Degree bounds for semi-invariant rings of quivers ☆

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ABSTRACT

We use recent results on matrix semi-invariants to give degree bounds on generators for the ring of semi-invariants for quivers with no oriented cycles.

1. Introduction

In [3], we studied the left-right action of $\mathrm{SL}_n \times \mathrm{SL}_n$ on m-tuples of $n \times n$ matrices. Among other things, we proved bounds for the degree of generators for the invariant ring. This ring of invariants can be seen as the ring of semi-invariants for the m-Kronecker quiver for the dimension vector (n, n). In this paper, we obtain bounds for the degree of generators for the ring of semi-invariants for a quiver with no oriented cycles.

1.1. Quiver representations

A quiver is just a directed graph. Formally, a quiver Q is a pair (Q_0, Q_1), where Q_0 is a finite set of vertices, and Q_1 is a finite set of arrows. For each arrow $a \in Q_1$, we denote its head and tail by ha and ta respectively. We fix an infinite field K. A representation V of Q over K is a collection of finite dimensional K-vector spaces $V(x)$, $x \in Q_0$ together with a collection of K-linear maps $V(a) : V(ta) \to V(ha)$, $a \in Q_1$. The dimension vector of V is the function $\alpha : Q_0 \to \mathbb{Z}_{\geq 0}$ such that $\alpha(x) = \dim V(x)$ for all $x \in Q_0$. We say a dimension vector is sincere if $\alpha(x) \neq 0 \; \forall \; x \in Q_0$.

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Let $\mathrm{Mat}_{p,q}$ denote the set of $p \times q$ matrices over K. For a dimension vector $\alpha \in \mathbb{Z}_{\geq 0}^{Q_0}$, we define its representation space by:

$$\mathrm{Rep}(Q, \alpha) = \prod_{a \in Q_1} \mathrm{Mat}_{\alpha(ha),\alpha(ta)}.$$

If V is a representation with dimension vector α and we identify $V(x) \cong K^{\alpha(x)}$ for all x, then V can be viewed as an element of $\mathrm{Rep}(Q, \alpha)$. Consider the group $\mathrm{GL}(\alpha) = \prod_{x \in Q_0} \mathrm{GL}_{\alpha(x)}$ and its subgroup $\mathrm{SL}(\alpha) = \prod_{x \in Q_0} \mathrm{SL}_{\alpha(x)}$. The group $\mathrm{GL}(\alpha)$ acts on $\mathrm{Rep}(Q, \alpha)$ by:

$$(A(x) \mid x \in Q_0) \cdot (V(a) \mid a \in Q_1) = (A(ha)V(a)A(ta)^{-1} \mid a \in Q_1).$$

For $V \in \mathrm{Rep}(Q, \alpha)$, choosing a different basis means acting by the group $\mathrm{GL}(\alpha)$. The $\mathrm{GL}(\alpha)$-orbits in $\mathrm{Rep}(Q, \alpha)$ correspond to isomorphism classes of representations of dimension α.

1.2. Invariants for quiver representations

The group GL_n acts by simultaneous conjugation on $\mathrm{Mat}_{n,n}^m$, the space of m-tuples of $n \times n$ matrices. Artin conjectured that in characteristic 0, the invariant ring is generated by traces of words in the matrices. Procesi proved this conjecture, and also showed that invariants of degree $\leq 2^n - 1$ generate the ring of invariants (see [23]). It was shown by Razmyslov that invariants of degree $\leq n^2$ suffice (see [24, final remark]). A concise account of the above results can also be found in [9].

Le Bruyn and Procesi generalized the results to arbitrary quivers. They proved that the ring of invariants $K[\mathrm{Rep}(Q, \alpha)]^{\mathrm{GL}(\alpha)}$ is generated by traces along oriented cycles. Using the aforementioned bound, they showed that the invariants of degree $\leq N^2$ generate the ring, where $N = \sum_i \alpha_i$ (see [19]).

1.3. Semi-invariants for quiver representations

From Le Bruyn and Procesi's results described above, we see that a quiver with no oriented cycles has no non-trivial invariants. However, the ring of semi-invariants $\mathrm{SI}(Q, \alpha) = K[\mathrm{Rep}(Q, \alpha)]^{\mathrm{SL}(\alpha)}$ could still be interesting.

A multiplicative character of the group GL_α is of the form

$$\chi_\sigma : (A(x) \mid x \in Q_0) \in \mathrm{GL}_\alpha \mapsto \prod_{x \in Q_0} \det(A(x))^{\sigma(x)} \in K^\star,$$

where $\sigma : Q_0 \to \mathbb{Z}$ is called the weight of the character χ_σ. Define

$$\mathrm{SI}(Q, \alpha)_\sigma = \{f \in K[\mathrm{Rep}(Q, \alpha)] \mid \forall A \in \mathrm{GL}(\alpha)\ A \cdot f = \chi_\sigma(A) f\}.$$

Then the ring of semi-invariants has a weight space decomposition

$$\mathrm{SI}(Q, \alpha) = \bigoplus_\sigma \mathrm{SI}(Q, \alpha)_\sigma.$$

If $\sigma \cdot \alpha = \sum_{x \in Q_0} \sigma(x)\alpha(x) \neq 0$, then $\mathrm{SI}(Q, \alpha)_\sigma = 0$. Assume that $\sigma \cdot \alpha = 0$. We can write $\sigma = \sigma_+ - \sigma_-$ where $\sigma_+(x) = \max\{\sigma(x), 0\}$ and $\sigma_-(x) = \max\{-\sigma(x), 0\}$. Define $|\sigma|_\alpha = \sigma_+ \cdot \alpha = \sigma_- \cdot \alpha$.

Now we define an $n \times n$ linear matrix

$$A : \bigoplus_{x \in Q_0} V(x)^{\sigma_+(x)} \to \bigoplus_{x \in Q_0} V(x)^{\sigma_-(x)}$$