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Current superalgebras and unitary representations

Karl-Hermann Neeb [a,1], Malihe Yousofzadeh [b,c,*,2]

[a] *Department Mathematik, FAU, Cauerstrasse 11, 91058 Erlangeen, Germany*
[b] *Department of Mathematics, University of Isfahan, P.O. Box 81745-163, Isfahan, Iran*
[c] *School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran*

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ABSTRACT

In this paper we determine the projective unitary representations of finite dimensional Lie supergroups whose underlying Lie superalgebra is $\mathfrak{g} = A \otimes \mathfrak{k}$, where $\mathfrak{k}$ is a compact simple Lie superalgebra and A is a supercommutative associative (super)algebra; the crucial case is when $A = \Lambda_s(\mathbb{R})$ is a Graßmann algebra. Since we are interested in projective representations, the first step consists in determining the cocycles defining the corresponding central extensions. Our second main result asserts that, if $\mathfrak{k}$ is a simple compact Lie superalgebra with $\mathfrak{k}_1 \neq \{0\}$, then each (projective) unitary representation of $\Lambda_s(\mathbb{R}) \otimes \mathfrak{k}$ factors through a (projective) unitary representation of $\mathfrak{k}$ itself, and these are known by Jakobsen's classification. If $\mathfrak{k}_1 = \{0\}$, then we likewise reduce the classification problem to semidirect products of compact Lie groups K with a Clifford–Lie supergroup which has been studied by Carmeli, Cassinelli, Toigo and Varadarajan.

1. Introduction

In a similar fashion as projective unitary representations $\pi\colon G \to \mathrm{PU}(\mathscr{H})$ of a Lie group G implement symmetries of quantum systems modeled on a Hilbert space $\mathscr{H}$, projective unitary representations of Lie supergroups implement symmetries of super-symmetric quantum systems [2]. Here the Hilbert space is replaced by a super Hilbert space $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$, i.e., a direct sum of two subspaces corresponding to a $\mathbb{Z}_2$-grading of $\mathscr{H}$. We deal with Lie supergroups as Harish–Chandra pairs $\mathcal{G} = (G, \mathfrak{g})$, where G is a Lie group and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra, where $\mathfrak{g}_0$ is the Lie algebra of G, and we have an *adjoint action* of G on $\mathfrak{g}$ by automorphisms of the Lie superalgebra $\mathfrak{g}$ extending the adjoint action of G on its Lie algebra $\mathfrak{g}_0$.

* Corresponding author.
E-mail addresses: neeb@math.fau.de (K.-H. Neeb), ma.yousofzadeh@sci.ui.ac.ir, ma.yousofzadeh@ipm.ir (M. Yousofzadeh).
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The concept of a unitary representation of a Lie supergroup $\mathcal{G}$ consists of a unitary representation π of G by grading preserving unitary operators and a representation χ_π of the Lie superalgebra $\mathfrak{g}$ on the dense subspace $\mathscr{H}^\infty$ of smooth vectors for G such that natural compatibility conditions are satisfied (see Definition 2.3 for details). To accommodate the fact that the primary interest lies in projective unitary representations, one observes that projective representations lift to unitary representations of central extensions by the circle group $\mathbb{T}$ acting on $\mathscr{H}$ by scalar multiplication. Having this in mind, one can directly study unitary representations of central extensions (see [6] for more details on this passage).

The corresponding classification problem splits into two layers. One is to determine the even central extensions of a given Lie supergroup $\mathcal{G}$ and the second consists of determining for these central extensions the corresponding unitary representations.

The existence of an invariant measure implies that for any finite dimensional Lie group G, unitary representations exist in abundance, in particular the natural representation on $L^2(G)$ is injective. This is drastically different for Lie supergroups, for which all unitary representations may be trivial. The reason for this is that, for every unitary representation $\chi\colon \mathfrak{g} \to \mathrm{End}(\mathscr{H}^\infty)$ and every odd element $x_1 \in \mathfrak{g}_1$, the operator $-i\chi([x_1,x_1])$ is non-negative. This imposes serious positivity restrictions on the representations on the even part $\mathfrak{g}_0$, namely that $-i\chi(x) \geq 0$ for all elements in the closed convex cone $\mathscr{C}(\mathfrak{g}) \subseteq \mathfrak{g}_0$ generated by all brackets $[x_1,x_1]$ of odd elements. Accordingly, $\mathfrak{g}$ has no faithful unitary representation if the cone $\mathscr{C}(\mathfrak{g})$ is not pointed (cf. [9]). Put differently, the kernel of any unitary representation contains the ideal $\mathrm{urad}(\mathfrak{g})$ of the Lie superalgebra $\mathfrak{g}$ generated by the linear subspace $\mathfrak{E} := \mathscr{C}(\mathfrak{g}) \cap -\mathscr{C}(\mathfrak{g})$ of $\mathfrak{g}_0$ and all those elements $x \in \mathfrak{g}_1$ with $[x,x] \in \mathfrak{E}$.

A particularly simple but nevertheless important class of Lie superalgebras are the *Clifford–Lie superalgebras* $\mathfrak{g}$ for which $[\mathfrak{g}_0,\mathfrak{g}] = \{0\}$ ($\mathfrak{g}_0$ is central), so that the Lie bracket of $\mathfrak{g}$ is determined by a symmetric bilinear map $\mu\colon \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$. If $\mathfrak{g}_0 = \mathbb{R}$ and the symmetric bilinear form μ is indefinite, then $\mathfrak{g}$ has no non-zero unitary representations.

In [1] the authors have determined the structure of finite dimensional Lie superalgebras $\mathfrak{g}$ for which finite dimensional unitary representations exist. This property implies in particular that $\mathfrak{g}$ is *compact* in the sense that $e^{\mathrm{ad}\mathfrak{g}_0} \subseteq \mathrm{Aut}(\mathfrak{g})$ is a compact subgroup, but, unlike the purely even case, this condition is not sufficient for the existence of finite dimensional projective unitary representations. In particular, it is shown in [1] that only four families of simple compact Lie superalgebras have finite dimensional projective unitary representations: $\mathfrak{su}(n|m;\mathbb{C}), n \neq m$, $\mathfrak{psu}(n|n;\mathbb{C})$, $\mathfrak{c}(n)$ and $\mathfrak{pq}(n)$ (see Subsection 4.2 for details).

In this paper we take the next step by considering *current Lie superalgebras* $\mathfrak{g} = A\otimes\mathfrak{k}$, where $\mathfrak{k}$ is a compact simple Lie superalgebra and A is a supercommutative associative (super)algebra and study the projective unitary representations, resp., the unitary representations of central extensions of these Lie superalgebras. Since we are interested in the phenomena caused by the superstructure, the main interest lies in algebras A generated by their odd part A_1. As the supercommutativity implies that the squares of odd elements in A vanish, any such A is a quotient of a Graßmann algebra. Therefore the main point is to understand current superalgebras of the form $\Lambda_s(\mathbb{R}) \otimes \mathfrak{k}$, where $\mathfrak{k}$ is a compact simple Lie superalgebra and $\Lambda_s(\mathbb{R})$ is the Graßmann algebra with s generators.

Our main result are the following. As explained below, we first have to understand the structure of the central extensions, resp., of the even 2-cocycles. This is described in Section 3 and works as follows. Suppose that κ is a non-degenerate invariant symmetric bilinear form on $\mathfrak{k}$ which is invariant under all derivations of $\mathfrak{k}$. Then any $D \in \mathrm{der}(\mathfrak{k})$ and any linear map $f\colon A \to \mathbb{R}$ leads to a 2-cocycle

$$\eta_{f,D}(a \otimes x, b \otimes y) := (-1)^{|b||x|} f(ab)\kappa(Dx, y).$$

There is a second class of natural cocycles on $A \otimes \mathfrak{k}$. To describe it, we call a bilinear map $F\colon A \times A \longrightarrow \mathbb{R}$ a *Hochschild map* if

$$F(a,b) = -(-1)^{|a||b|}F(b,a) \quad \text{and} \quad F(ab,c) = F(a,bc) + (-1)^{|b||a|}F(b,ac)$$