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Differential smoothness of skew polynomial rings

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ABSTRACT

It is shown that, under some natural assumptions, the tensor product of differentially smooth algebras and the skew-polynomial rings over differentially smooth algebras are differentially smooth.

1. Introduction

The study of smoothness of algebras goes back at least to Grothendieck's EGA. The concept of a formally smooth commutative (topological) algebra introduced in there [5, Définition 19.3.1] was later extended to the non-commutative case by Schelter in [12]. An algebra is formally smooth if and only if the kernel of the multiplication map is projective as a bimodule. As argued by Schelter himself, this notion arose as a replacement of a far too general definition based on the finiteness of the global dimension. Although it plays an important role in non-commutative geometry (see e.g. [4], where such algebras are termed *quasi-free*), the notion of formal smoothness seems to be too restrictive. The too crude notion of smoothness based on the finiteness of the global dimensions was refined in [13], where a Noetherian algebra was said to be smooth provided that it had a finite global dimension equal to the homological dimension of all its simple modules. From the homological perspective probably most satisfying is the notion of *homological smoothness* introduced in [14]: an algebra is homologically smooth provided it admits a finite resolution by finitely generated projective bimodules. Algebras of this kind display a Poincaré type duality between Hochschild homology and cohomology, and retain many properties characteristic of co-ordinate algebras of smooth varieties (see e.g. [8], where this last point is strongly argued for).

A different and more constructive approach to smoothness of algebras was taken in [3]. In this approach the smoothness of an algebra A is related to the existence of a specific differential graded algebra (with A as

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the degree-zero part) whose size is aligned with the rate of growth of A measured by the Gelfand–Kirillov dimension, and which satisfies a strict version of the Poincaré duality in terms of an isomorphism with the corresponding complex of integral forms [2] (see Section 2 for precise definition). In view of this direct use of differential graded algebras this kind of smoothness is referred to as *differential smoothness*. The main advantage of this approach is its concreteness: a differentially smooth algebra comes equipped with a well-behaved differential structure and with the precisely defined concept of integration. Examples of differentially smooth algebras include the coordinate algebras of the quantum group $SU_q(2)$, the quantum 2-sphere (see [2]), the non-commutative pillow algebra, the quantum cone algebras (see [3]), the quantum polynomial algebras (see [6]), and Hopf algebra domains of Gelfand–Kirillov dimension 2 that are not PI (see [1]). Although many of these examples are known to be also homologically smooth, the relationship between the differential and other types of smoothness is not clear yet.

At the root of difficulties with comparing differential and other types of smoothness is the constructive nature of the former, which prevents one from using functorial or just existential arguments. In this paper we make a few steps toward resolving such difficulties and present two general constructions which lead from differentially smooth to differentially smooth algebras. First, we show that – under some natural assumptions on differential structures and algebras – the tensor product of differentially smooth algebras is differentially smooth. This allows one to deduce quickly smoothness of polynomial and Laurent polynomial rings without necessity of constructing a specific differential structure (it suffices to have such a structure for polynomials in one variable). Second, again under some natural assumptions, we prove that the skew-polynomial rings over a smooth algebra are smooth.

2. Preliminaries

Let $\mathbb{F}$ be a field. By a *differential calculus* over an $\mathbb{F}$-algebra R we mean a differential graded algebra $(\Omega R, d)$ (i.e. a graded algebra with the degree-one square-zero linear map $d : \Omega R \to \Omega R$ satisfying the graded Leibniz rule) such that:

(a) $\Omega R = \bigoplus_{n\in\mathbb{N}} \Omega^n R$, i.e. it is non-negatively graded, and $\Omega^0 R = R$,
(b) For all $n \in \mathbb{N}$,

$$\Omega^n R = R\,\underbrace{d(R)d(R)\cdots d(R)}_{n\text{-times}}.$$

The requirement (b) is called the *density condition*. A differential calculus $(\Omega R, d)$ over R is said to be *connected*, provided $\ker(d\,|_R) = \mathbb{F}$. It is said to have *dimension* N or to be N*-dimensional* provided

$$\Omega^N R \neq 0 \qquad \text{and} \qquad \Omega^n R = 0, \quad \text{for all } n > N.$$

An N-dimensional differential calculus $(\Omega R, d)$ over R is said to *admit a volume form*, provided $\Omega^N R$ is isomorphic to R as both a left and right R-module (but not necessarily as an R-bimodule). Any free generator v of $\Omega^N R$ as a right and left R-module is referred to as a *volume form*. Associated to a volume form v are two maps:

(a) the right R-module *co-ordinate isomorphism*:

$$\pi_{\mathsf{v}} : \Omega^N R \to R, \qquad \pi_{\mathsf{v}}(\mathsf{v}r) = r; \tag{2.1}$$

(b) the *twisting* algebra automorphism:

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