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Bordered surfaces in the 3-sphere with maximum symmetry

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A R T I C L E I N F O

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A B S T R A C T

We consider orientation-preserving actions of finite groups G on pairs (S^3, Σ), where Σ denotes a compact connected surface embedded in S^3. In a previous paper, we considered the case of closed, necessarily orientable surfaces, determined for each genus $g > 1$ the maximum order of such a G for all embeddings of a surface of genus g, and classified the corresponding embeddings.

In the present paper we obtain analogous results for the case of bordered surfaces Σ (i.e. with non-empty boundary, orientable or not). Now the genus g gets replaced by the algebraic genus α of Σ (the rank of its free fundamental group); for each $\alpha > 1$ we determine the maximum order m_α of an action of G, classify the topological types of the corresponding surfaces (topological genus, number of boundary components, orientability) and their embeddings into S^3. For example, the maximal possibility $12(\alpha - 1)$ is obtained for the finitely many values $\alpha = 2, 3, 4, 5, 9, 11, 25, 97, 121$ and 241.

1. Introduction

We study smooth, faithful actions of finite groups G on pairs (S^3, Σ) where Σ denotes a compact, connected, bordered surface with an embedding $e : \Sigma \to S^3$ (so G is a finite group of diffeomorphisms of a pair (S^3, Σ)). We also say that such a G-action on Σ is *extendable* (w.r.t. e).

Let $\Sigma_{g,b}$ denote the orientable compact surface of (topological) genus g with b boundary components, writing also Σ_g instead of $\Sigma_{g,0}$; for $g > 0$, let $\Sigma^-_{g,b}$ denote the non-orientable compact surface of genus g with b boundary components. $\Sigma^-_{g,b}$ is obtained from the connected sum of g real projective planes by creating b boundary components (by deleting the interiors of b disjoint embedded disks), and it is well-known that each compact surface is either $\Sigma_{g,b}$ or $\Sigma^-_{g,b}$.

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For $b > 0$, $\Sigma_{g,b}$ and $\Sigma^-_{g,b}$ are bordered surfaces, and we use $\alpha(\Sigma)$ to denote their algebraic genus equal to the rank of the free fundamental group $\pi_1(\Sigma)$; this is also the genus of a regular neighborhood of Σ in S^3 which is a 3-dimensional handlebody. We have $\alpha(\Sigma_{g,b}) = 2g - 1 + b$ and $\alpha(\Sigma^-_{g,b}) = g - 1 + b$. We will always assume that $\alpha > 1$ in the present paper.

We will consider only orientation-preserving finite group actions on S^3; then, referring to the recent geometrization of finite group actions on S^3 (see [2] for the case of non-free actions and [10] for the general case), we can restrict to orthogonal actions of finite groups on S^3, i.e. to finite subgroups G of the orthogonal group $SO(4)$.

Let m_α denote the maximum order of such a group G acting on a pair (S^3, Σ), for all embeddings of bordered surfaces Σ of a fixed algebraic genus α into S^3. In the present paper we will determine m_α and classify all surfaces Σ which realize the maximum order m_α.

A similar question for the pair (S^3, Σ_g), where Σ_g is the closed orientable surface of genus g, was studied in [12]. The corresponding maximum order OE_g of finite groups acting on (S^3, Σ_g) for all possible embeddings $\Sigma_g \subset S^3$ was obtained in that paper.

Let V_g denote the handlebody of genus g. Each bordered surface $\Sigma \subset S^3$ of algebraic genus α has a regular neighborhood which is homeomorphic to V_α. We note that, similar as for handlebodies, the maximal possibilities for the orders of groups of homeomorphisms of compact bordered surfaces of algebraic genus α are $12(\alpha-1)$, $8(\alpha-1)$, $20(\alpha-1)/3$, $6(\alpha-1)$, ... (see section 3 of [9]), and these are exactly the values occurring in the next theorem. A classification of all finite group actions on compact bordered surfaces up to algebraic genus 101 is given in [1], and the lists in that paper may be compared with the list in the next theorem. Concerning other papers considering symmetries of surfaces immersed in 3-space, see [3], [4] and [8].

Our main result is:

Theorem 1.1. *For each $\alpha > 1$, m_α and the surfaces realizing m_α are listed below.*

α	m_α	Σ
2	$12(\alpha-1) = 12$	$\Sigma_{0,3}$, $\Sigma_{1,1}$
3	$12(\alpha-1) = 24$	$\Sigma_{0,4}$, $\Sigma^-_{1,3}$
4	$12(\alpha-1) = 36$	$\Sigma_{1,3}$
5	$12(\alpha-1) = 48$	$\Sigma_{0,6}$, $\Sigma_{1,4}$
9	$12(\alpha-1) = 96$	$\Sigma_{2,6}$, $\Sigma_{3,4}$
11	$12(\alpha-1) = 120$	$\Sigma_{0,12}$, $\Sigma^-_{6,6}$
25	$12(\alpha-1) = 288$	$\Sigma_{7,12}$, $\Sigma_{10,6}$
97	$12(\alpha-1) = 1152$	$\Sigma_{37,24}$
121	$12(\alpha-1) = 1440$	$\Sigma_{43,36}$, $\Sigma_{55,12}$
241	$12(\alpha-1) = 2880$	$\Sigma_{73,96}$, $\Sigma_{97,48}$, $\Sigma^-_{206,36}$
7	$8(\alpha-1) = 48$	$\Sigma_{0,8}$, $\Sigma^-_{4,4}$
49	$8(\alpha-1) = 384$	$\Sigma_{17,16}$, $\Sigma_{21,8}$
16	$\frac{20}{3}(\alpha-1) = 100$	$\Sigma_{6,5}$
19	$\frac{20}{3}(\alpha-1) = 120$	$\Sigma_{0,20}$, $\Sigma^-_{14,6}$
361	$\frac{20}{3}(\alpha-1) = 2400$	$\Sigma_{131,100}$, $\Sigma_{151,60}$, $\Sigma_{171,20}$
21	$6(\alpha-1) = 120$	$\Sigma_{5,12}$
481	$6(\alpha-1) = 2880$	$\Sigma_{205,72}$, $\Sigma_{193,96}$
41	$\frac{24}{5}(\alpha-1) = 192$	$\Sigma^-_{30,12}$
1681	$\frac{30}{7}(\alpha-1) = 7200$	$\Sigma^-_{1562,120}$
841	$4(\sqrt{\alpha}+1)^2 = 3600$	$\Sigma_{391,60}$, $\Sigma_{406,30}$
k^2, $k \neq 3, 5, 7, 11, 19, 41$	$4(\sqrt{\alpha}+1)^2$	$\Sigma_{\frac{k(k-1)}{2},k+1}$
29	$4(\alpha+1) = 120$	$\Sigma_{0,30}$, $\Sigma_{9,12}$, $\Sigma_{14,2}$
the remaining numbers	$4(\alpha+1)$	$\Sigma_{0,\alpha+1}$, $\Sigma_{\frac{\alpha}{2},1}$ (α even), $\Sigma_{\frac{\alpha-1}{2},2}$ (α odd)

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