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A tight bound on the projective dimension of four quadrics

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A B S T R A C T

Motivated by Stillman's question, we show that the projective dimension of an ideal generated by four quadric forms in a polynomial ring is at most 6; moreover, this bound is tight. We achieve this bound, in part, by giving a characterization of the low degree generators of ideals primary to height three primes of multiplicities one and two.

1. Introduction

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k, and let $I = (f_1, \ldots, f_N)$ be a homogeneous ideal of S. Classical theorems of Hilbert or Auslander and Buchsbaum give upper bounds on the projective dimension of S/I (or, equivalently, of I) in terms of n, the number of variables of S. Motivated by computational efficiency issues, Stillman [23] posed the following question:

Question 1.1 *(Stillman [23, Problem 3.14]).* Is there a bound on $\operatorname{pd}(S/I)$ depending only on $d_1, \ldots, d_N$, and N, where $d_i = \deg(f_i)$?

Ananyan and Hochster [1] recently proved an affirmative answer to Stillman's Question in full generality. They prove that for any homogeneous ideal I generated by N forms of degree at most d either the generators form a regular sequence, implying that N is an upper bound for the projective dimension, or else the generators of I are contained in an ideal generated by a bounded number (in terms of N and d) of forms of

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strictly smaller degree. A delicate inductive proof shows the existence of a bound, but even in the case when I is generated by quadrics, the bounds they produce are very large and far from optimal. Tighter bounds in more specific cases have been given in the following situations:

1. In a previous paper [2], Ananyan and Hochster showed that if I is generated by N quadric forms, then $\mathrm{pd}(S/I)$ has an upper bound asymptotic to $2N^{2N}$. They also give more general result for non-homogeneous ideals generated in degree at most 2. In [1, Theorem 4.2] they claim a new upper bound of $2^{N+1}(N-2)+4$.
2. When I is minimally generated by N quadrics and $\mathrm{ht}(I) = 2$, we previously showed [15] that $\mathrm{pd}(S/I) \leq 2N-2$. (See Theorem 2.12.) Moreover, for all N we construct examples where equality is achieved. Thus the bound here is optimal.
3. When I is minimally generated by 3 cubics, Engheta [12] showed that $\mathrm{pd}(S/I) \leq 36$, while the largest known example satisfies $\mathrm{pd}(S/I) = 5$.
4. When $\mathrm{k} = p > 0$ and S/I is an F-pure ring, De Stefani and Núñez-Betancourt [7] showed that $\mathrm{pd}(S/I) \leq \mu(I)$, where $\mu(I)$ denotes the minimal number of generators of I.

It remains open as to what the optimal bounds for projective dimension are. The general optimal bound, however, must be rather large. Examples of Beder et. al. [5] show that the projective dimension of ideals generated by 3 degree-d forms can grow exponentially with respect to d.

Further motivating Question 1.1, Caviglia showed it was equivalent to the following analogous question for Castelnuovo–Mumford regularity. (See [20, Theorem 2.4].)

Question 1.2 *(Stillman [23, Problem 3.15]).* Is there a bound on $\mathrm{reg}(S/I)$ depending only on $d_1, \ldots, d_N$, and N, where $d_i = \deg(f_i)$?

While (1) above shows that $\mathrm{pd}(S/I)$ is bounded for any ideal generated by N quadrics, these bounds are exponential in N. It is clear already in the case of an ideal generated by 3 quadrics that these bounds are far from optimal (cf. [20, Proposition 24]). In this paper we give a tight upper bound on the projective dimension of an ideal generated by 4 quadrics. Specifically, we prove

Theorem 1.3. *Let S by a polynomial ring over a field k, and let $I = (q_1, q_2, q_3, q_4)$ be an ideal of S generated by 4 homogeneous polynomials of degree 2. Then $\mathrm{pd}(S/I) \leq 6$.*

Examples from [19] of ideals I generated by 4 quadrics with $\mathrm{pd}(S/I) = 6$ show that the above bound is optimal. Our proof follows a technique similar to Engheta's in [10] and [12] in dividing into cases by height and multiplicity. We notably rely on the tight bound for the height 2 case proved in [15] and the height 3 multiplicity 6 case proved in [16]. The remaining height 3 cases constitute the bulk of this paper and require a variety of approaches.

Previously the authors had posted a paper to the arXiv proving an upper bound of 9; however, we suspected that 6 should be the tight upper bound. We finally reduced all of the remaining cases to 6 producing this paper. This was done at the expense of the length and complexity of the paper. However, we do not believe these arguments can be significantly shortened without weakening the upper bound and, along with Theorem 2.12, this paper represents the only other nontrivial known tight bound for Stillman's Question. A table summarizing the many cases is given in Section 10.

The rest of the paper is organized as follows: in Section 2, we collect many of the results that we cite and fix notation; in Section 3, we prove some results that are used several times throughout the paper; in Sections 4, 5, we characterize certain height three primary ideals that can occur as the unmixed part of an ideal of four quadrics or a direct link of such an ideal; in Sections 6, 7, 8, we handle the cases where our

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