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Decomposition of integer-valued polynomial algebras

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ABSTRACT

Let D be a commutative domain with field of fractions K, let A be a torsion-free D-algebra, and let B be the extension of A to a K-algebra. The set of integer-valued polynomials on A is $\mathrm{Int}(A) = \{f \in B[X] \mid f(A) \subseteq A\}$, and the intersection of $\mathrm{Int}(A)$ with $K[X]$ is $\mathrm{Int}_K(A)$, which is a commutative subring of $K[X]$. The set $\mathrm{Int}(A)$ may or may not be a ring, but it always has the structure of a left $\mathrm{Int}_K(A)$-module.

A D-algebra A which is free as a D-module and of finite rank is called Int_K-decomposable if a D-module basis for A is also an $\mathrm{Int}_K(A)$-module basis for $\mathrm{Int}(A)$; in other words, if $\mathrm{Int}(A)$ can be generated by $\mathrm{Int}_K(A)$ and A. A classification of such algebras has been given when D is a Dedekind domain with finite residue rings. In the present article, we modify the definition of Int_K-decomposable so that it can be applied to D-algebras that are not necessarily free by defining A to be Int_K-decomposable when $\mathrm{Int}(A)$ is isomorphic to $\mathrm{Int}_K(A) \otimes_D A$. We then provide multiple characterizations of such algebras in the case where D is a discrete valuation ring or a Dedekind domain with finite residue rings. In particular, if D is the ring of integers of a number field K, we show that an Int_K-decomposable algebra A must be a maximal D-order in a separable K-algebra B, whose simple components have as center the same finite unramified Galois extension F of K and are unramified at each finite place of F. Finally, when both D and A are rings of integers in number fields, we prove that Int_K-decomposable algebras correspond to unramified Galois extensions of K.

1. Introduction

Let D be a commutative integral domain with field of fractions K. The ring of integer-valued polynomials over D is defined to be $\mathrm{Int}(D) := \{f \in K[X] \mid f(D) \subseteq D\}$. The ring $\mathrm{Int}(D)$, its elements, and its properties have been popular objects of study over the past several decades and continue to be so today. The book [4] is the standard reference on the topic.

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Beginning around 2010, attention turned to polynomials that are evaluated on D-algebras rather than on D itself. This can be seen in the work of Evrard, Fares and Johnson [6,7], Frisch [8–11], Loper [15], Peruginelli [5,13,20–23,25], Werner [30,32,33], and Naghipour, Rismanchian, and Sedighi Hafshejani [17]. A good example of these new rings of integer-valued polynomials comes from considering the polynomials in $K[X]$ that map each element of the matrix algebra $M_n(D)$ back to $M_n(D)$.

Example 1.1. Associate K with the scalar matrices in $M_n(K)$. Then, for any matrix $a \in M_n(D)$ and any polynomial $f(X) = \sum_{i=0}^{t} q_i X^i \in K[X]$, we can evaluate f at a to produce the matrix $f(a) = \sum_{i=0}^{t} q_i a^i$. If $f(a) \in M_n(D)$ for each $a \in M_n(D)$, then f is said to be integer-valued on $M_n(D)$. The set of all such polynomials is denoted by

$$\mathrm{Int}_K(M_n(D)) := \{f \in K[X] \mid f(M_n(D)) \subseteq M_n(D)\},$$

and it is easy to verify that $\mathrm{Int}_K(M_n(D))$ is a subring of $K[X]$.

We can form a larger collection of polynomials that are integer-valued on $M_n(D)$ by considering polynomials whose coefficients come from $M_n(K)$ rather than from K. That is, we form the set

$$\mathrm{Int}(M_n(D)) := \{f \in M_n(K)[X] \mid f(M_n(D)) \subseteq M_n(D)\}.$$

Since $M_n(K)$ is noncommutative, we follow standard conventions regarding polynomials with noncommuting coefficients, as in [14, §16]. In $M_n(K)[X]$, we assume that the indeterminate X commutes with each element of $M_n(K)$, and we define evaluation to occur when the indeterminate is to the right of any coefficients. So, given $f(X) = \sum_{i=0}^{t} q_i X^i \in M_n(K)[X]$, we consider $f(X)$ to be equal to $\sum_{i=0}^{t} X^i q_i$ as an element of $M_n(K)[X]$, but to evaluate $f(X)$ at a matrix $a \in M_n(D)$, we must first write $f(X)$ in the form $f(X) = \sum_{i=0}^{t} q_i X^i$, and then $f(a) = \sum_{i=0}^{t} q_i a^i$. A consequence of this is that evaluation is no longer a multiplicative homomorphism; that is, if $f(X) = g(X)h(X)$ in $M_n(K)[X]$, then it may not be true that $f(a)$ equals $g(a)h(a)$. Because of this difficulty, it is not clear whether $\mathrm{Int}(M_n(D))$ is closed under multiplication. Despite the complications associated with evaluation of polynomials in this setting, one may prove that $\mathrm{Int}(M_n(D))$ is a (noncommutative) subring of $M_n(K)[X]$ [31, Thm. 1.2]. Thus, we are able to construct a noncommutative ring of integer-valued polynomials.

We can actually say more. In [8, Thm. 7.2], Sophie Frisch proved that $\mathrm{Int}(M_n(D))$ is itself a matrix ring. Specifically, $\mathrm{Int}(M_n(D)) \cong M_n(\mathrm{Int}_K(M_n(D)))$, where the isomorphism is given by associating a polynomial with matrix coefficients to a matrix with polynomial entries. (This isomorphism is the restriction of the classical isomorphism between the polynomial ring $M_n(K)[X]$ and the matrix ring $M_n(K[X])$.) Because of Frisch's theorem, many questions about $\mathrm{Int}(M_n(D))$ can be reduced to questions about $\mathrm{Int}_K(M_n(D))$, and the latter ring—being commutative—is usually easier to work with.

Broadly speaking, the point of this paper is to study the relationship between a commutative ring of integer-valued polynomials such as $\mathrm{Int}_K(M_n(D))$ and its extension $\mathrm{Int}(M_n(D))$. In particular, we wish to determine when and how Frisch's theorem [8, Thm. 7.2] can be generalized to algebras other than matrix rings. While matrix rings will be prominent in our work, the majority of our theorems deal with general algebras. However, our basic definitions are inspired by the situation described in Example 1.1.

We begin by giving notation and conventions for working with polynomials over algebras. As before, let D be a commutative integral domain with field of fractions K. Let A be a torsion-free D-algebra and take $B = K \otimes_D A$ to be the extension of A to a K-algebra. We associate K and A with their canonical images in B via the maps $k \mapsto k \otimes 1$ and $a \mapsto 1 \otimes a$. Much of our work will involve polynomials in $B[X]$. The algebra B may be noncommutative, but we will assume that X commutes with all elements of B. Moreover, we define evaluation of polynomials in $B[X]$ at elements of A just as we did in Example 1.1 where $A = M_n(D)$ and $B = M_n(K)$. Given $f(X) = \sum_{i=0}^{t} c_i X^i \in B[X]$ and $b \in B$, we define