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A local duality principle in derived categories of commutative Noetherian rings

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ABSTRACT

Let R be a commutative Noetherian ring. We introduce the notion of colocalization functors γ_W with supports in arbitrary subsets W of $\operatorname{Spec} R$. If W is a specialization-closed subset, then γ_W coincides with the right derived functor $\mathrm{R}\Gamma_W$ of the section functor Γ_W with support in W. We prove that the local duality theorem and the vanishing theorem of Grothendieck type hold for γ_W with W being an arbitrary subset.

1. Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring. We denote by $\mathcal{D} = D(\operatorname{Mod} R)$ the derived category of complexes of R-modules, by which we mean that $\mathcal{D}$ is the unbounded derived category. Neeman [14] proved that there is a natural one-one correspondence between the set of subsets of $\operatorname{Spec} R$ and the set of localizing subcategories of $\mathcal{D}$. We denote by $\mathcal{L}_W$ the localizing subcategory corresponding to a subset W of $\operatorname{Spec} R$. The localization theory of triangulated categories [11] yields a right adjoint γ_W to the inclusion functor $\mathcal{L}_W \hookrightarrow \mathcal{D}$, and such an adjoint is unique. This functor $\gamma_W : \mathcal{D} \to \mathcal{L}_W (\hookrightarrow \mathcal{D})$ is our main target of this paper, and we call it *the colocalization functor with support in* W.

If V is a specialization-closed subset of $\operatorname{Spec} R$, then γ_V is nothing but the right derived functor $\mathrm{R}\Gamma_V$ of the section functor Γ_V with support in V, whose ith right derived functor $H^i_V(-) = H^i(\mathrm{R}\Gamma_V(-))$ is known as the ith local cohomology functor. For a general subset W of $\operatorname{Spec} R$, the colocalization functor γ_W is not necessarily a right derived functor of an additive functor defined on the category $\operatorname{Mod} R$ of R-modules.

In this paper, we establish several results concerning the colocalization functor γ_W, where W is an arbitrary subset of $\operatorname{Spec} R$. Notable are extensions of the local duality theorem and Grothendieck type vanishing theorem of local cohomology. The local duality can be viewed as an isomorphism

$$\mathrm{R}\Gamma_V \operatorname{RHom}_R(X, Y) \cong \operatorname{RHom}_R(X, \mathrm{R}\Gamma_V Y),$$

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where V is a specialization-closed subset of $\operatorname{Spec} R$, $X \in \mathcal{D}^-_{\mathrm{fg}}$ and $Y \in \mathcal{D}^+$; see [5, Proposition 6.1]. The following theorem generalizes this isomorphism to the case of colocalization functors γ_W.

Theorem 1.1 *(Theorem 4.1). Let W be a subset of $\operatorname{Spec} R$ and let $X, Y \in \mathcal{D}$. We denote by $\dim W$ the supremum of the lengths of chains of prime ideals in W. Suppose that one of the following conditions holds:*

(1) *$X \in \mathcal{D}^-_{\mathrm{fg}}$, $Y \in \mathcal{D}^+$ and $\dim W$ is finite;*
(2) *$X \in \mathcal{D}_{\mathrm{fg}}$, Y is a bounded complex of injective R-modules and $\dim W$ is finite;*
(3) *W is generalization-closed.*

Then there exists a natural isomorphism

$$\gamma_W \operatorname{RHom}_R(X, Y) \cong \operatorname{RHom}_R(X, \gamma_W Y).$$

We shall call Theorem 1.1 the *Local Duality Principle*, which naturally implies the following corollary.

Corollary 1.2 *(Corollary 4.5). Assume that R admits a dualizing complex D_R. Let W be an arbitrary subset of $\operatorname{Spec} R$ and $X \in \mathcal{D}_{\mathrm{fg}}$. We write $X^\dagger = \operatorname{RHom}_R(X, D_R)$. Then we have a natural isomorphism*

$$\gamma_W X \cong \operatorname{RHom}_R(X^\dagger, \gamma_W D_R).$$

The local duality theorem states the validity of this isomorphism in the case that W is specialization-closed, see [8, Chapter V; Theorem 6.2] and [5, Corollary 6.2].

As an application of the Local Duality Principle, we can prove the vanishing theorem of Grothendieck type for the colocalization functor γ_W with support in an arbitrary subset W. Let $\mathfrak{a}$ be an ideal of R and $X \in \mathcal{D}$. The $\mathfrak{a}$-depth of X, which we denote by $\operatorname{depth}(\mathfrak{a}, X)$, is the infimum of the set $\{i \in \mathbb{Z} \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, X) \neq 0\}$. More generally, for a specialization-closed subset W, the W-depth of X, which we denote by $\operatorname{depth}(W, X)$, is defined as the infimum of the set of values $\operatorname{depth}(\mathfrak{a}, X)$ for all ideals $\mathfrak{a}$ with $V(\mathfrak{a}) \subseteq W$. When $X \in \mathcal{D}_{\mathrm{fg}}$, we denote by $\dim X$ the supremum of the set $\{\dim H^i(X) + i \mid i \in \mathbb{Z}\}$.

For a finitely generated R-module M, the Grothendieck vanishing theorem says that the ith local cohomology module $H^i_W(M) = H^i(\mathrm{R}\Gamma_W M)$ of M with support in W is zero for $i < \operatorname{depth}(W, M)$ and $i > \dim M$. We are able to generalize this theorem to the following result in §6.

Theorem 1.3 *(Theorem 6.5). Assume that R admits a dualizing complex. Let W be an arbitrary subset of $\operatorname{Spec} R$ with the specialization closure $\overline{W}^s$. If $X \in \mathcal{D}_{\mathrm{fg}}$, then $H^i(\gamma_W X) = 0$ unless $\operatorname{depth}(\overline{W}^s, X) \leq i \leq \dim X$.*

In §3, we give an explicit description of γ_W for subsets W of certain special type, see Theorem 3.12. For example, if W is a one-point set $\{\mathfrak{p}\}$, then it is proved the colocalization functor $\gamma_{\{\mathfrak{p}\}}$ equals $\mathrm{R}\Gamma_{V(\mathfrak{p})} \operatorname{RHom}_R(R_\mathfrak{p}, -)$, see Corollary 3.3. This is one of the rare cases that we know the explicit form of γ_W, while for a general subset W we give in Theorem 3.13 the way how we calculate γ_W by the induction on $\dim W$.

In §4, we give a complete proof of the Local Duality Principle (Theorem 1.1).

The subsequent section §5 is devoted to the relationship between γ_W and left derived functors of completion functors. In particular, we see that there is a subset W such that $H^i(\gamma_W I) \neq 0$ for an injective module I and some $i < 0$. This observation shows that γ_W is not a right derived functor of an additive functor defined on $\operatorname{Mod} R$ in general.

In the last section §6, we present a precise and complete proof for Theorem 1.3 above.

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