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On the stable hom relation and stable degenerations of Cohen–Macaulay modules ☆

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ABSTRACT

We study the stable hom relation for Cohen–Macaulay modules over Gorenstein local algebras. We give the sufficient condition to make the stable hom relation a partial order when the base algebra is of finite representation type. As an application, we give the description of stable degenerations of Cohen–Macaulay modules over simple singularities of several types by using the stable hom relation.

1. Introduction

In ring theory, the hom relation is a basic relation for finitely generated modules over finite dimensional algebras [1,2,12,18,15,9]. Let k be a field and R a k-algebra. The relation is defined by a dimension of a hom-set between finitely generated modules as a k-module, that is, we define the relation $M \leq_{hom} N$ by a relation $\dim_k \mathrm{Hom}_R(X, M) \leq \dim_k \mathrm{Hom}_R(X, N)$ for each finitely generated modules X. Auslander–Reiten [2] use the relation to investigate when indecomposable modules are determined by the composition factors. In [18], Zwara gave a characterization of degenerations of modules over representation finite algebras in relation with the hom relation. We remark that the hom relation is not always a partial order. It has been studied by many authors [1–3,15] when the relation is actually a partial order.

In the paper, we investigate the hom relation on a stable category of Cohen–Macaulay modules $\underline{\mathrm{CM}}(R)$ over (not necessary Artinian) Gorenstein k-algebra. First we compare the Auslander–Reiten theory on $\mathrm{CM}(R)$ with that on $\underline{\mathrm{CM}}(R)$. We look into the relation between AR sequences and AR triangles of Cohen–Macaulay modules (Proposition 2.2). We consider a relation on $\underline{\mathrm{CM}}(R)$ which is the stable analogue of the hom relation (Definition 2.5) and shall show that it is actually a partial order if the algebra is of finite

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representation type with certain assumptions (Theorem 2.9). In Section 4, we attempt to characterize the stable degenerations of Cohen–Macaulay modules by using the stable hom relation. The concept of stable degenerations of Cohen–Macaulay modules was introduced by Yoshino [17]. It is closely related to ordinary degenerations of modules [7,13]. We shall show that the stable degenerations over several simple singularities can be controlled by the stable hom relation (Theorem 4.6). To show this, we use the stable analogue of the argument over finite dimensional algebras in [18]. As a conclusion, we give the description of stable degenerations of Cohen–Macaulay modules over simple singularities of type (A_n) (Theorem 4.15).

The stable hom relation has been studied by Auslander–Reiten [2] and they also considered when the relation is a partial order. But the techniques in this paper are different from them because they used the fact that the ordinary (not stable) hom relation is a partial order. In our setting, the hom-set does not always have finite dimension, so that we can not apply their argument.

2. Stable hom relation on Cohen–Macaulay modules

Throughout the paper R is a commutative complete Gorenstein local k-algebra where k is an algebraically closed field of characteristic 0. For a finitely generated R-module M, we say that M is a Cohen–Macaulay R-module if

$$\mathrm{Ext}^i_R(M, R) = 0 \quad \text{for any } i > 0.$$

We denote by $\mathrm{CM}(R)$ the category of Cohen–Macaulay R-modules with all R-homomorphisms. We also denote by $\underline{\mathrm{CM}}(R)$ the stable category of $\mathrm{CM}(R)$. The objects of $\underline{\mathrm{CM}}(R)$ are the same as those of $\mathrm{CM}(R)$, and the morphisms of $\underline{\mathrm{CM}}(R)$ are elements of $\underline{\mathrm{Hom}}_R(M, N) = \mathrm{Hom}_R(M, N)/P(M, N)$ for $M, N \in \underline{\mathrm{CM}}(R)$, where $P(M, N)$ denote the set of morphisms from M to N factoring through free R-modules. We write $\underline{\mathrm{Hom}}_R(M, N)$ for $\mathrm{Hom}_{\underline{\mathrm{CM}}(R)}(M, N)$. For a Cohen–Macaulay module M, denote by $\underline{M}$ to indicate that it is an object of $\underline{\mathrm{CM}}(R)$. For a finitely generated R-module M, take a free resolution

$$\cdots \to F_1 \xrightarrow{d} F_0 \to M \to 0.$$

We denote $\mathrm{Im}(d)$ by ΩM. We also denote $\mathrm{Coker}(\mathrm{Hom}_R(d, R))$ by $\mathrm{Tr}M$, which is called an Auslander transpose of M. We note that the functor Ω defines a functor giving an auto-equivalence on $\underline{\mathrm{CM}}(R)$. It is known that $\underline{\mathrm{CM}}(R)$ has a structure of a triangulated category with the suspension functor defined by the functor Ω. See [6, Chapter 1] for details. Since R is Gorenstein, by the definition of a triangle, $\underline{L} \to \underline{M} \to \underline{N} \to \underline{L}[1]$ is a triangle in $\underline{\mathrm{CM}}(R)$ if and only if there is an exact sequence $0 \to L \to M' \to N \to 0$ in $\mathrm{CM}(R)$ with $\underline{M'} \cong \underline{M}$ in $\underline{\mathrm{CM}}(R)$, that is, M' is isomorphic to M up to free summand. Since R is complete, $\mathrm{CM}(R)$, hence $\underline{\mathrm{CM}}(R)$, is a Krull–Schmidt category, namely each object can be decomposed into indecomposable objects up to isomorphism uniquely.

In the paper we use the theory of Auslander–Reiten (abbr. AR) sequences and triangles of Cohen–Macaulay modules. Let us recall the definitions of those notions. See [14] for AR sequences and [6,11] for AR triangles.

Definition 2.1. Let X, Y and Z be Cohen–Macaulay R-modules.

(1) A short exact sequence $\Sigma_X : 0 \to Z \to Y \xrightarrow{f} X \to 0$ is said to be an AR sequence ending in X (or starting from Z) if it satisfies
   - (AR1) X and Z are indecomposable.
   - (AR2) Σ_X is not split.
   - (AR3) If $g : W \to X$ is not a split epimorphism, then there exists $h : W \to Y$ such that $g = f \circ h$.

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