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Algebras whose right nucleus is a central simple algebra

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ABSTRACT

We generalize Amitsur's construction of central simple algebras over a field F which are split by field extensions possessing a derivation with field of constants F to nonassociative algebras: for every central division algebra D over a field F of characteristic zero there exists an infinite-dimensional unital nonassociative algebra whose right nucleus is D and whose left and middle nucleus are a field extension K of F splitting D, where F is algebraically closed in K.

We then give a short direct proof that every p-algebra of degree m, which has a purely inseparable splitting field K of degree m and exponent one, is a differential extension of K and cyclic. We obtain finite-dimensional division algebras over a field F of characteristic $p > 0$ whose right nucleus is a division p-algebra.

0. Introduction

In 1954, Amitsur [2] observed that all associative central division algebras over a field F of characteristic zero can be constructed using differential polynomials. His construction method can be considered as an analogue to the well known crossed product construction, except that he uses splitting fields K of the algebras, where the base field F is algebraically closed in K, instead of their algebraic splitting fields. Some of his results also work for p-algebras, i.e. over base fields of characteristic $p > 0$.

In this paper, we consider algebras which are also obtained from differential polynomials, but which are nonassociative.

These algebras are constructed using the differential polynomial ring $K[t;\delta]$, where K is a field and δ a derivation on K and were defined by Petit [13]: given a differential polynomial $f \in K[t;\delta]$ of degree m, the set of all differential polynomials of degree less than m, together with the addition given by the usual addition of polynomials, can be equipped with a nonassociative ring structure using right division by f to define the multiplication as $g \circ h = gh \bmod_r f$. The resulting nonassociative unital ring S_f, also denoted by $K[t;\delta]/K[t;\delta]f$, is an algebra over the field of constants $F = \mathrm{Const}(\delta)$ of δ. If f generates a two-sided ideal

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in $K[t;\delta]$, then S_f is the (associative) quotient algebra obtained by factoring out the two-sided principal ideal generated by f.

If f is not two-sided and δ not trivial, then the nuclei of S_f are larger than the center $F = \mathrm{Const}(\delta)$. In that case the left and middle nucleus are always given by K, whereas the right nucleus reflects both the choice of f and the structure of the ring $K[t;\delta]$.

We proceed as follows: The basic terminology and notation we use can be found in [2] and Section 1. Section 2 rephrases some of Amitsur's results for those algebras S_f which have a central simple algebra as their right nucleus. For this we employ Amitsur's A-polynomials. In Sections 3 and 4 we show how to construct algebras S_f with a given central simple algebra as right nucleus, first for base fields of characteristic zero, then for base fields of characteristic $p > 0$: for every central simple algebra B of degree m over a field F of characteristic zero which is split by a field extension K/F in which F is algebraically closed, there exists an infinite-dimensional unital algebra $S_f = K[t;\delta]/K[t;\delta]f$ over F with right nucleus B (and left and middle nucleus K), see Theorem 8. In particular, for every central division algebra D over F there exists an infinite-dimensional unital algebra S_f over F with right nucleus D (Corollary 9).

We present a short proof that every p-algebra B of degree m over a field F of characteristic p which is split by a purely inseparable field extension K/F of exponent one and degree m is isomorphic to a differential extension (K, δ, d_0) of K (Theorem 13), only invoking a result on the structure of S_f and Amitsur's [2, Lemma 20']. Thus it is cyclic by [8, Main Theorem].

For every division p-algebra D of degree m over a field F of characteristic p which is split by a purely inseparable field extension K/F of exponent one such that $m < [K : F]$, there is a unital division algebra $S_f = K[t;\delta]/K[t;\delta]f$ over F of dimension mp^e with right nucleus D and left and middle nucleus K. The smallest possible dimension l of such a division algebra containing D as right nucleus is bounded via $m^2 < l \leq mp^{m-1}$ and connected to the number of cyclic algebras that are needed when expressing D as a product of cyclic algebras of degree p in the Brauer group $Br(F)$ (Corollary 18).

1. Preliminaries

1.1. Nonassociative algebras

Let F be a field and let A be an F-vector space. A is an *algebra* over F if there exists an F-bilinear map $A \times A \to A$, $(x, y) \mapsto x \cdot y$, denoted simply by juxtaposition xy, the *multiplication* of A. An algebra A is called *unital* if there is an element in A, denoted by 1, such that $1x = x1 = x$ for all $x \in A$. We will only consider unital algebras from now on without explicitly saying so.

An algebra $A \neq 0$ is called a *division algebra* if for any $a \in A$, $a \neq 0$, the left multiplication with a, $L_a(x) = ax$, and the right multiplication with a, $R_a(x) = xa$, are bijective. If A has finite dimension over F, A is a division algebra if and only if A has no zero divisors [16, pp. 15, 16].

Associativity in A is measured by the *associator* $[x, y, z] = (xy)z - x(yz)$. The *left nucleus* of A is defined as $\mathrm{Nuc}_l(A) = \{x \in A \,|\, [x, A, A] = 0\}$, the *middle nucleus* of A as $\mathrm{Nuc}_m(A) = \{x \in A \,|\, [A, x, A] = 0\}$ and the *right nucleus* of A as $\mathrm{Nuc}_r(A) = \{x \in A \,|\, [A, A, x] = 0\}$. $\mathrm{Nuc}_l(A)$, $\mathrm{Nuc}_m(A)$, and $\mathrm{Nuc}_r(A)$ are associative subalgebras of A. Their intersection $\mathrm{Nuc}(A) = \{x \in A \,|\, [x, A, A] = [A, x, A] = [A, A, x] = 0\}$ is the *nucleus* of A. $\mathrm{Nuc}(A)$ is an associative subalgebra of A containing $F1$ and $x(yz) = (xy)z$ whenever one of the elements x, y, z is in $\mathrm{Nuc}(A)$. The *center* of A is $\mathrm{C}(A) = \{x \in \mathrm{Nuc}(A) \,|\, xy = yx \text{ for all } y \in A\}$.

1.2. Differential polynomial rings

Let K be a field and $\delta : K \to K$ a *derivation*, i.e. an additive map such that

$$\delta(ab) = a\delta(b) + \delta(a)b$$