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Tautological classes on the moduli space of hyperelliptic curves with rational tails

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A B S T R A C T

We study tautological classes on the moduli space of stable n-pointed hyperelliptic curves of genus g with rational tails. The method is based on the approach of Yin in comparing tautological classes on the moduli of curves and the universal Jacobian. Our result gives a complete description of tautological relations. It is proven that all relations come from the Jacobian side. The intersection pairings are shown to be perfect in all degrees. We show that the tautological algebra coincides with its image in cohomology via the cycle class map. The latter is identified with monodromy invariant classes in cohomology.

0. Introduction

The study of tautological classes on spaces of curves was initiated by Mumford in his seminal article [23]. Tautological classes are natural algebraic cycles reflecting the nature of the generic object parameterized by the moduli space. The set of generators consists of an explicit collection of cycles. Tautological groups are finite dimensional vector spaces. This distinguishes remarkably the tautological ring from the out of reach space of all algebraic cycles. A basic question about tautological algebras is to give a meaningful class of relations among tautological classes. In [5] Faber introduced a method to produce tautological relations on the moduli space $\mathcal{M}_g$ of smooth curves of genus $g > 2$. He conjectured that his method produces all tautological relations and the resulting algebra is Gorenstein. Analogous conjectures were formulated for spaces of pointed curves by Faber and Pandharipande. See [6] for a survey. There are counterexamples for the Gorenstein conjectures on $\overline{\mathcal{M}}_{2,20}$ and $\mathcal{M}^{ct}_{2,8}$ [27,29]. However, for the spaces of curves with rational tails they are still open. More recent conjectures are Yin's conjecture for the universal curve $\mathcal{C}_g$ over $\mathcal{M}_g$ [41] and Pixton's conjecture [30] for the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$. These conjectures are wide open in full generality. The tautological rings of $\mathcal{M}_g$ for $g < 24$ were computed by Faber. In genus 24

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all known methods fail to give a Gorenstein algebra. The analysis of the tautological ring of the universal curve $\mathcal{C}_g$ for $g < 20$ is due to Yin. From his conjecture in [41] the tautological ring of $\mathcal{C}_{20}$ should not be Gorenstein. See also the tables by Pixton [31] for more cases that his conjecture contradicts the Gorenstein conjecture. Tautological rings of pointed spaces have been computed in genus at most two [17,27,25,38,36,37]. See also [39] for computations in higher genus. In our previous work [37] we studied the tautological ring of the moduli space $\mathcal{M}^{rt}_{2,n}$. Our analysis was based on a series of tautological relations discovered by Faber, Getzler, Belorousski and Pandharipande [4,8,2]. A new relation of degree 3 involving 6 points was also found. It was shown that these relations lead to a complete description of tautological relations on $\mathcal{M}^{rt}_{2,n}$ for every n. We were not able to give a uniform proof of these relations and in each case the proofs were involved.

Pixton's relations [30] are conjectured to produce all relations on $\overline{\mathcal{M}}_{g,n}$. However there is no general method to produce tautological relations on the space of special curves. The goal of this article is to analyse the tautological ring in the simplest case of hyperelliptic curves. The study of more general cases is the subject of upcoming articles. Our study is based on the thesis of Yin [41] who studied the connection between tautological classes on moduli of curves and the universal Jacobian. Tautological classes on both parts were studied extensively. Nevertheless their connection was not understood before his work. In his analysis Yin studied tautological classes on the universal curve $\mathcal{C}_g$ over $\mathcal{M}_g$. Here we consider the space $\mathcal{H}^{rt}_{g,n}$ which classifies stable n-pointed hyperelliptic curves of genus g with rational tails. We show that all tautological relations can be easily obtained from relations on the universal Jacobian $\mathcal{J}_g$ over the space $\mathcal{H}_g$. The reduced fiber of $\mathcal{H}^{rt}_{g,n}$ over a moduli point $[X]$ associated with a smooth hyperelliptic curve X is the Fulton–MacPherson compactification $X[n]$ of the configuration space of n points on X introduced in [7]. The tautological ring of the fiber $X[n]$ is defined naturally by restriction. We will show the following statement:

Theorem 0.1. *Let X be a fixed hyperelliptic curve of genus g. The tautological ring of the moduli space $\mathcal{H}^{rt}_{g,n}$ is naturally isomorphic to the tautological ring of the fiber $X[n]$.*

From Theorem 0.1 we obtain a complete presentation of the tautological ring in terms of generators and relations. It also follows that the tautological ring has the Gorenstein property. This means that the tautological ring has a form of Poincaré duality. Our proof gives a uniform proof of the previous result in genus two as well as for hyperelluptic curves in all genera. Everything mentioned above concerns tautological classes in Chow. The Gorenstein property of $R^*(\mathcal{H}^{rt}_{g,n})$ implies the same results in cohomology. Using a result of Petersen and Tommasi, which was our motivation for this project, we prove the following:

Corollary 0.2. *The cycle class map induces an isomorphism between the tautological ring of the moduli space $\mathcal{H}^{rt}_{g,n}$ in Chow and monodromy invariant classes in cohomology.*

Conventions 0.3. We work over an algebraically closed field of characteristic different from 2, 3. We consider algebraic cycles modulo rational equivalence. Chow rings and cohomology rings are taken with $\mathbb{Q}$-coefficients.

1. Tautological classes on the space of hyperelliptic curves

Let $\mathcal{H}^{rt}_{g,n}$ be the space of stable n-pointed hyperelliptic curves of genus $g \geq 2$ with rational tails for a natural number n. Recall that $(C; x_1, \ldots, x_n)$ is said to be an n-pointed stable curve of genus g if it satisfies the following conditions: the curve C is a nodal curve of arithmetic genus g and the markings x_i, for $1 \leq i \leq n$, belong to its smooth locus. Nodes and the markings on each irreducible component of C are called special points. The stability condition implies that all rational components have at least 3 special points. Such a curve is said to have *rational tails* if exactly one of its irreducible components has geometric genus g. The coarse moduli space associated to $\mathcal{H}_g := \mathcal{H}^{rt}_{g,0}$ is a quasi-projective variety of dimension $2g-1$