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# On the spectrum of rings of functions ☆

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ABSTRACT

For $D$ a domain and $E \subseteq D$, we investigate the prime spectrum of rings of functions from $E$ to $D$, that is, of rings contained in $\prod_{e\in E} D$ and containing $D$. Among other things, we characterize, when $M$ is a maximal ideal of finite index in $D$, those prime ideals lying above $M$ which contain the kernel of the canonical map to $\prod_{e\in E}(D/M)$ as being precisely the prime ideals corresponding to ultrafilters on $E$. We give a sufficient condition for when all primes above $M$ are of this form and thus establish a correspondence to the prime spectra of ultraproducts of residue class rings of $D$. As a corollary, we obtain a description using ultrafilters, differing from Chabert's original one which uses elements of the $M$-adic completion, of the prime ideals in the ring of integer-valued polynomials $\mathrm{Int}(D)$ lying above a maximal ideal of finite index.



## 1. Introduction

Let $D$ be an integral domain, $E \subseteq D$, and $\mathcal{R}$ a subring of $\prod_{e\in E} D$, containing $D$. The elements of $\mathcal{R}$ can be interpreted as functions from $E$ to $D$ and, consequently, we call $\mathcal{R}$ a ring of functions from $E$ to $D$. We will investigate the prime spectra of such rings of functions. We obtain, for quite general $\mathcal{R}$, a partial description of the prime spectrum, cf. Theorems 3.7 and 5.3, and in special cases a complete characterization, cf. Corollary 6.5.

Our motivation is the spectrum of a ring of integer-valued polynomials: For $D$ an integral domain with quotient field $K$, let $\mathrm{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}$ be the ring of integer-valued polynomials on $D$. More generally, when $K$ is understood, we let $\mathrm{Int}(A, B) = \{f \in K[x] \mid f(A) \subseteq B\}$ for $A, B \subseteq K$.

If $D$ is a Noetherian one-dimensional domain, a celebrated theorem of Chabert [1, Ch. V] states that every prime ideal of $\mathrm{Int}(D)$ lying over a maximal ideal $M$ of finite index in $D$ is maximal and of the form

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$$M_\alpha = \{f \in \mathrm{Int}(D) \mid f(\alpha) \in \hat{M}\},$$

where $\alpha$ is an element of the $M$-adic completion $\hat{D}_M$ of $D$ and $\hat{M}$ the maximal ideal of $\hat{D}_M$.

In fact, Chabert showed two separate statements independently − both under the assumption that $D$ is Noetherian and one-dimensional and $M$ a maximal ideal of finite index of $D$:

(1) Every maximal ideal of $\mathrm{Int}(D)$ containing $\mathrm{Int}(D, M)$ is of the form $M_\alpha$ for some $\alpha \in \hat{D}_M$.
(2) Every maximal ideal of $\mathrm{Int}(D)$ lying over $M$ contains $\mathrm{Int}(D, M)$.

For a simplified proof of Chabert's result, see [4], Lemma 4.4 and the remark following it.

We will show that a modified version of statement (1) holds in far greater generality, for rings of functions. The modification consists in replacing elements of the $M$-adic completion by ultrafilters.

Whether (2) holds or not for a particular $D$ and a particular subring of $D^E$ will have to be examined separately. It is, in some sense, a question of density of the subring in the product $\prod_{e\in E} D$.

We will work in the following setting:

**Definition 1.1.** Let $D$ be a commutative ring and $E \subseteq D$. Let $\mathcal{R}$ be a commutative ring and $\varphi\colon \mathcal{R} \to \prod_{e\in E} D$ a monomorphism of rings. $\varphi$ allows us to interpret the elements of $\mathcal{R}$ as functions from $E$ to $D$.

If all constant functions are contained in $\varphi(\mathcal{R})$, we call the pair $(\mathcal{R}, \varphi)$ a ring of functions from $E$ to $D$. We use $\mathcal{R} = \mathcal{R}(E, D)$ (where $\varphi$ is understood) to denote a ring of functions from $E$ to $D$.

**Remark 1.2.** For our considerations it is vital that $\mathcal{R} = \mathcal{R}(E, D)$ contain all constant functions, because we will make extensive use of the following fact: when $\mathcal{I}$ is an ideal of $\mathcal{R} = \mathcal{R}(E, D)$, $f \in \mathcal{I}$ and $g \in D[x]$ a polynomial with zero constant term, then $g(f) \in \mathcal{I}$, and similarly, if $g$ is a polynomial in several variables over $D$ with zero constant term, and an element of $\mathcal{I}$ is substituted for each variable in $g$, then, an element of $\mathcal{I}$ results.

Let us note that considerable research has been done on the spectrum of a power of a ring $D^E = \prod_{e\in E} D$ or a product of rings $\prod_{e\in E} D_e$. Gilmer and Heinzer [5, Prop. 2.3] have determined the spectrum of an infinite product of local rings, and Levy, Loustaunau and Shapiro [8] that of an infinite power of $\mathbb{Z}$. Our focus here is not on the full product of rings, but on comparatively small subrings and the question of how much information about the spectrum of a ring can be obtained from its embedding in a power of a domain.

One ring can be embedded in different products: $\mathrm{Int}(D)$ can be seen as a ring of functions from $K$ to $K$ as well as a ring of functions from $D$ to $D$. We will glean a lot more information about the spectrum of $\mathrm{Int}(D)$ from the second interpretation than from the first.

## 2. Prime ideals corresponding to ultrafilters

Let $\mathcal{R} = \mathcal{R}(E, D)$ be a ring of functions from $E$ to $D$ as in Definition 1.1. We will now make precise the concept of ideals corresponding to ultrafilters, and the connection to ultraproducts $\prod_{e\in E}^{\mathcal{U}}(D/M)$, where $M$ is a maximal ideal of $D$, and $\mathcal{U}$ an ultrafilter on $E$. First a quick review of filters, ultrafilters and ultraproducts:

**Definition 2.1.** Let $S$ be a set. A non-empty collection $\mathcal{F}$ of subsets of $S$ is called a filter on $S$ if

(1) $\emptyset \notin \mathcal{F}$.
(2) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.
(3) $A \subseteq C \subseteq S$ with $A \in \mathcal{F}$ implies $C \in \mathcal{F}$.

A filter $\mathcal{F}$ on $S$ is called an ultrafilter on $S$ if, for every $C \subseteq S$, either $C \in \mathcal{F}$ or $S \setminus C \in \mathcal{F}$.

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