Journal of Pure and Applied Algebra 222 (2018) 2099–2142

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

A categorical approach to the maximum theorem

Seerp Roald Koudenburg

Mathematics Research and Teaching Group, Middle East Technical University, Northern Cyprus Campus, 99738 Kalkanlı, Güzelyurt, TRNC, via Mersin 10, Turkey

A R T I C L E I N F O

Article history:
Received 1 April 2017
Received in revised form 10 August 2017
Communicated by J. Adámek

MSC:
49J27; 49J45; 54A05; 54B30; 54E70; 18A40; 18C20; 18D05

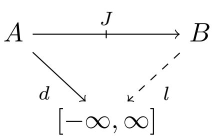
A B S T R A C T

Berge's maximum theorem gives conditions ensuring the continuity of an optimised function as a parameter changes. In this paper we state and prove the maximum theorem in terms of the theory of monoidal topology and the theory of double categories.

This approach allows us to generalise (the main assertion of) the maximum theorem, which is classically stated for topological spaces, to pseudotopological spaces and pretopological spaces, as well as to closure spaces, approach spaces and probabilistic approach spaces, amongst others. As a part of this we prove a generalisation of the extreme value theorem.

0. Introduction

Berge's maximum theorem [3], which is used in mathematical economics for instance, concerns a relation $J\colon A \nrightarrow B$ between topological spaces, which we regard as a subset $J \subseteq A \times B$, as well as a continuous map $d\colon A \to [-\infty, \infty]$ into the extended real line, as depicted in the following diagram.



We may "extend d along J" by "optimising d for each $y \in B$", thus obtaining a map $l\colon B \to [-\infty, \infty]$ given by the suprema

$$l(y) = \sup_{x \in J^\circ y} d(x), \tag{1}$$

E-mail address: roald@metu.edu.tr.

http://dx.doi.org/10.1016/j.jpaa.2017.09.002

where $J^\circ y = \{x \in A \mid (x, y) \in J\}$ denotes the preimage of y under J. The main assertion of the maximum theorem states that the optimised function l is continuous as soon as the relation J is 'hemi-continuous' and $J^\circ y \neq \emptyset$ for each $y \in B$. Among the conditions included in hemi-continuity is the compactness of the preimages $J^\circ y$ so that, by the extreme value theorem, hemi-continuity of J implies that the suprema defining l are attained as maxima—a consequence that is used in the classical proof of the maximum theorem.

Regarding the ordered set $[-\infty, \infty]$ as a category allows us to think of the suprema in (1) as being limits. In fact, we may consider the full optimised function l as the 'left Kan extension of d along J', a construction that is fundamental to category theory. Recently it has been shown that, mostly in purely categorical settings, structure on a 'morphism' $D\colon \mathcal{A} \to \mathcal{M}$ carries over to Kan extensions of D under certain conditions—the monoidal structure on a functor for instance, see [26], [20] and [33]. The maximum theorem can be thought of as fitting in the same scheme of results: it shows that the continuity of the map d carries over to its left Kan extension l. In view of this, one might hope to discover a purely categorical result that, in the topological setting, recovers the classical maximum theorem while, when considered in other settings, allows us to obtain generalisations of the maximum theorem. This paper realises this hope to a large extent.

Besides recognising optimised functions as Kan extensions, the second ingredient of our categorical approach to the maximum theorem is to regard topological structures as algebraic structures—a point of view that forms the basis of the study of 'monoidal topology' [17]. In the fundamental example for instance, one regards topologies on a set A as closure operations, i.e. relations $c\colon PA \nrightarrow A$ between the powerset PA of A and A itself: one defines $(S, x) \in c$ precisely if $x \in \bar{S}$, the closure of $S \subseteq A$. The axioms for a topology on A then translate to three axioms on the 'closure relation' c and, by weakening or removing some of these axioms, generalisations of the notion of topological space are recovered, such as that of pretopological space [4] and closure space.

The closure relation $c\colon PA \nrightarrow A$ above can be equivalently thought of as a map $c\colon PA \times A \to \{\bot, \top\}$ taking values in the set $\{\bot, \top\}$ of truth values. A second way of generalising the notion of topological space, which is fundamental to monoidal topology, is to replace the set of truth values by a different set of values $\mathcal{V}$. In this way for instance, by considering $[0, \infty]$-valued closure relations $\delta\colon PA \times A \to [0, \infty]$, one recovers the notion of approach space [25], consisting of a set A equipped with a point–set distance $\delta(S, x) \in [0, \infty]$ for each subset $S \subseteq A$ and point $x \in A$. Likewise, by allowing closure relations to take 'distance distribution functions' $\phi\colon [0, \infty] \to [0, 1]$ as values, one obtains the notion of probabilistic approach space [21].

Besides closure operations, the notion of topology can be described algebraically in terms of ultrafilter convergence as well [2]: topologies on a set A correspond precisely to convergence relations $\alpha\colon UA \nrightarrow A$ satisfying certain axioms, where UA denotes the set of ultrafilters on A. As with closure operations, by weakening these axioms, or by considering $\mathcal{V}$-valued convergence relations $\alpha\colon UA \times A \to \mathcal{V}$, one recovers generalisations of the notion of topological space, such as the notions of pretopological space and (probabilistic) approach space, as well as that of pseudotopological space [4], amongst others. In our approach to the maximal theorem we will consider both closure relations and ultrafilter convergence relations, as well as the relationship between them. In our study of the latter we closely follow [22].

The language allowing us to naturally describe the relations between the two ingredients of our approach—Kan extensions and algebraic descriptions of topological structures—is that of double categories, in the sense of e.g. [13]. The notion of double category extends that of category by considering two types of morphisms instead of the usual single type: e.g. between sets we will consider both functions $f\colon A \to C$ as well as $\mathcal{V}$-valued relations $J\colon A \times B \to \mathcal{V}$. Throughout this paper the language of double categories will lead us in the right direction. At the start for instance, when we consider approach spaces (equipped with $[0, \infty]$-valued closure relations), it naturally leads us to consider Kan extensions that are 'weighted' by $[0, \infty]$-valued relations $J\colon A \times B \to [0, \infty]$, instead of Kan extensions along ordinary relations $J\colon A \nrightarrow B$ as described above. Later it naturally leads to the generalisation of the notion of hemi-continuous relation, as well as to the proper generalisation of Kan extensions "whose suprema are attained by maxima". Finally, the language of