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Representation-finite Birkhoff type problems for nilpotent linear operators

Daniel Simson

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland

A R T I C L E I N F O

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In memoriam: to Sergei Adamovich Ovsienko (1.05.1953–25.01.2016)

A B S T R A C T

The paper is an addition to our article Simson (2015) [37]. Given a field K, an integer $m \geq 1$, and a finite poset $I \equiv (I, \preceq)$ with a unique maximal element $*$, we study the category $\mathcal{M}on(I, F_m)$ of mono-representations $U = (U_j)_{j\in I}$ of I over the Frobenius K-algebra $F_m := K[t]/(t^m)$ of K-dimension $m < \infty$. We view U as K-vector spaces U_*, with an m-nilpotent K-linear operator $\mathbf{t} : U_* \to U_*$, together with $\mathbf{t}$-invariant subspaces $U_i \subseteq U_j \subseteq U_*$, for all $i \preceq j \preceq *$ in I. The problem of when the Krull–Schmidt K-category $\mathcal{M}on(I, F_m)$ is of wild (resp. tame, finite) representation type is called a representation-wild (resp.-tame, -finite) Birkhoff type problem for m-nilpotent operators. In case when the field K is algebraically closed, we give in our previous paper a complete solution of the problem by describing all pairs (I, m), with $m \geq 2$, such that category $\mathcal{M}on(I, F_m)$ is representation-tame and representation-infinite. Moreover, the tame-wild dichotomy for the category $\mathcal{M}on(I, F_m)$ is proved there.

In the present paper all representation-finite Birkhoff type problems are explicitly described by means of the pairs (I, m). In particular case when $I = I_{a,b}$ is the union of two incomparable chains I' and I'' of length $|I'| = a-1 \geq 1$ and $|I''| = b-1 \geq 1$, with $I' \cap I'' = \{*\}$, we study a functorial connection vect-$\mathbb{X}(\mathbf{p}) \to \mathcal{M}on(I_{a,b}, F_m)$, where vect-$\mathbb{X}(\mathbf{p})$ is the vector bundle subcategory of the category coh-$\mathbb{X}(\mathbf{p})$ of coherent sheaves over the weighted projective line $\mathbb{X}(\mathbf{p})$, for the weight triple $\mathbf{p} = (a, b, m)$, with $a, b, m \geq 2$, applied by Kussin–Lenzing–Meltzer (2013) [18], in relation with the hypersurface singularity $f = x_1^a + x_2^b + x_3^m$. In this case we show that $\mathcal{M}on(I_{a,b}, F_m)$ is representation-finite (resp. representation-tame, representation-wild) if and only if the orbifold Euler characteristic $\chi_{(a,b,m)} = \frac{1}{a} + \frac{1}{b} + \frac{1}{m} - 1$ of $\mathbb{X}(a, b, m)$ is positive (resp. non-negative, negative).

1. Introduction

Throughout this paper, we fix a field K, an integer $m \geq 1$, and we set

$$F_m := K[t]/(t^m) \tag{1.1}$$

E-mail address: simson@mat.umk.pl.

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Note that F_m is a commutative uniserial Frobenius K-algebra of K-dimension $m < \infty$, i.e., F_m is injective viewed as an F_m-module, and every indecomposable F_m-module is isomorphic to an ideal $(\bar{t}^j)$ in the unique composition series $F_m \supset (\bar{t}) \supset (\bar{t}^2) \supset \ldots \supset (\bar{t}^{m-1}) \supset (0)$ of F_m, where $\bar{t} \in F_m$ is the coset of $t \in K[t]$ modulo the ideal (t^m). In particular, $F_1 = K$.

By a poset $I \equiv (I, \preceq)$ we mean a partially ordered set I with respect to a partial order relation $\preceq$. We write $i \prec j$, if $i \prec j$ and $i \neq j$. Throughout we assume that I is a **one-peak poset**, that is, I has a unique maximal element (we denote it by $*$), see [31] and [40].

To get a module-theoretical interpretation of a matrix problem studied by Plachotnik [26], we have introduced in [32] the following definition.

Given a field K, an integer $m \geq 1$, and a one-peak finite poset $I \equiv (I, \preceq)$, we define the category $\mathcal{M}on(I, F_m)$ of **mono-representations** of I over the K-algebra $F_m = K[t]/(t^m)$ (or **filtered I-chains** of finite-dimensional modules over F_m) to be the category of all I-tuples $U = (U_j)_{j\in I}$, where U_* is a finite-dimensional F_m-module, $U_j \subseteq U_*$ is an F_m-submodule of U_* such that $U_i \subseteq U_j \subseteq U_*$, if the relation $i \preceq j \preceq *$ holds in I. In other words, $U = (U_j)_{j\in I}$ can be viewed as a K-vector space U_*, together with an m-nilpotent K-linear operator $\mathbf{t} : U_* \to U_*$ and $\mathbf{t}$-invariant subspaces $U_i \subseteq U_j \subseteq U_*$, for all $i \preceq j \preceq *$ in I.

An illustrative example is the two-flag poset $I = I_{a,b}$

$$I_{a,b}: \quad \begin{array}{ccccc} & & * & & \\ & \nearrow & & \nwarrow & \\ \circ_{a-1} \longrightarrow \circ_{a-2} \longrightarrow \ldots \longrightarrow \circ_2 & & & & \circ_{2'} \longleftarrow \ldots \longleftarrow \circ_{(b-2)'} \longleftarrow \circ_{(b-1)'} \end{array} \tag{1.2}$$

where $a \geq 2$, $b \geq 2$. In this case, the filtered I-chain $U = (U_j)_{j\in I}$ can be viewed as the two-flag

$$U: \quad \begin{array}{ccccc} & & U_* \circlearrowleft \mathbf{t} & & \\ & \nearrow & & \nwarrow & \\ U_{a-1} \subseteq U_{a-2} \subseteq \ldots \subseteq U_2 & & & & U_{2'} \supseteq \ldots \supseteq U_{(b-2)'} \supseteq U_{(b-1)'} \end{array}$$

of inclusions between finite-dimensional $\mathbf{t}$-invariant K-vector subspaces of the vector space $U_* = U_1$, endowed with the K-linear operator $\mathbf{t} : U_* \to U_*$ of **nilpotency degree** $m \geq 2$, i.e., $\mathbf{t}^m = 0$. Note that in case when $a \geq 2$ and $b = 2$, the two-flag poset $I_{a,2}$ is the one-flag poset

$$I_{a,2} : \circ_{a-1} \longrightarrow \circ_{a-2} \longrightarrow \ldots \longrightarrow \circ_2 \longrightarrow * \,.$$

In particular, $I_{3,2}$ is the chain $\circ_2 \!\to\! *$ of length two and the category $\mathcal{M}on(I_{3,2}, F_m)$ consists of the K-linear $\mathbf{t}$-invariant monomorphisms $U = (U_2 \subseteq U_* = U_1)$ studied by Ringel and Schmidmeier in [28,29].

A morphism from $U = (U_j)_{j\in I}$ to $U' = (U'_j)_{j\in I}$ is a K-linear map $f : U_* \to U'_*$ such that $f(\mathbf{t}(u)) = \mathbf{t}(f(u))$, for all $u \in U_*$, and $f(U_j) \subseteq U'_j$, for each $j \in I$. We introduce an exact structure in $\mathcal{M}on(I, F_m)$ by defining a short sequence $0 \longrightarrow U' \xrightarrow{f'} U \xrightarrow{f''} U'' \longrightarrow 0$ in $\mathcal{M}on(I, F_m)$ to be **exact** if, for each $j \in I$, the sequence $0 \longrightarrow U_j \xrightarrow{f'_j} U_j \xrightarrow{f''_j} U''_j \longrightarrow 0$ of K-vector spaces is exact, where f'_j and f''_j are the restriction of f' and f'' to U'_j and to U_j, respectively. One easily shows that $\mathcal{M}on(I, F_m)$ is an additive (not necessarily abelian) Krull–Schmidt K-category.

The category $\mathcal{M}on(I, F_m)$ is defined to be of **finite representation type** if the number of the isomorphism classes of the indecomposable objects in $\mathcal{M}on(I, F_m)$ is finite. Tame representation type and wild representation type are defined in a usual way, see [37].

Following [28,29], [35], [36], and in analogy with a problem studied by Birkhoff [6] for abelian groups (later developed in [2] and [27]), we introduce in [37] the following definition. The problem of when the Krull–Schmidt K-category $\mathcal{M}on(I, F_m)$ is of finite, tame, or wild representation type is defined to be a **Birkhoff type problem** for nilpotent K-linear operators of nilpotency degree $m \geq 1$.

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