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On torsion Kähler differential forms

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ABSTRACT

Let k be a field of characteristic zero. Let $\mathscr{V}$ be an integral affine k-variety. We prove that any torsion Kähler differential form $\omega \in \Omega^1_{\mathscr{V}/k}(\mathscr{V})$ vanishes at every formal germ of curve on $\mathscr{V}$. If $\mathscr{V}$ is an integral hypersurface of $\mathbf{A}^N_k$ $(N \geq 1)$, we also prove that the non-cylindrical singular points of $\mathscr{V}$ belong to the singular locus of any torsion Kähler differential forms.

1. Introduction

1.1. The algebraic notion of torsion in the context of Kähler differential forms has been enlighted by the famous Berger conjecture which links, in dimension one, singularities to the existence (locally) of torsion Kähler differential forms (see Remark 3.1). The aim of the present article is to establish various original properties of torsion Kähler differential forms on algebraic varieties in relation with the geometry of the considered varieties.

1.2. Let k be a field of characteristic zero. Let $\mathscr{V} = \mathrm{Spec}(k[X,Y]/\langle f\rangle)$ be an integral affine plane curve, $D = a\partial_X + b\partial_Y \in \mathrm{Der}_k(k[X,Y])$ be a k-derivation of $k[X,Y]$ and $\tilde{\omega}_D = bdX - adY \in \Omega^1_{k[X,Y]/k}$. A basic algebraic observation shows that the following assertion are equivalent:

- The k-derivation D verifies $D(f) \in \langle f\rangle$, in other words D induces a k-derivation on $\mathscr{V}$;
- For every field extension K of k, and every formal parametrization $\varphi \in K[[T]]^2$ of the equation $\{f = 0\}$, we have $\varphi^*\tilde{\omega}_D = 0$;
- The Kähler differential form $\tilde{\omega}_D$ induces a torsion element of $\Omega^1_{\mathscr{V}/k}(\mathscr{V})$.

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The first main problem of this article can be formulated as follows:

Question 1.3. *Does this (kind of) property hold in higher dimension?*

We positively answer to this question by the following theorem:

Theorem 1.4. *Let k be a field of characteristic zero. Let $\bar{k}$ be an algebraic closure of k. Let $\mathscr{V}$ be an integral affine k-scheme of finite type. Let $\omega \in \Omega^1_{\mathscr{V}/k}(\mathscr{V})$. Then, the following assertions are equivalent:*

(1) The Kähler differential form ω is torsion;
(2) For every reduced k-algebra A, for every morphism of k-schemes $\varphi \in \mathscr{V}(A[[T]])$, we have $\varphi^*\omega = 0$;
(3) For every k-algebra A which is a domain, for every morphism of k-schemes $\varphi \in \mathscr{V}(A[[T]])$, we have $\varphi^*\omega = 0$;
(4) For every field extension K of k, for every morphism of k-schemes $\varphi \in \mathscr{V}(K[[T]])$, we have $\varphi^*\omega = 0$;
(5) For every morphism of k-schemes $\varphi \in \mathscr{V}(\bar{k}[[T]])$, we have $\varphi^*\omega = 0$.

Contrarily to the very particular case of plane curves where the data define, geometrically, a foliation $\omega = 0$ which is both of dimension and codimension 1, the arguments in the proof of the general case can not be built on this property and must be broadly adapted accordingly. The idea of the proof presented here is based on an observation coming from the arguments used in the proof of the irreducibility theorem due to E. Kolchin in differential algebra (see section 6).

1.5. The second main result contained in the present article concerns the comparison between the singular locus of $\mathscr{V}$ and the locus of $\mathscr{V}$ formed by the points on which a torsion Kähler differential form vanishes. If $\mathscr{V} = \mathrm{Spec}(k[X,Y]/\langle f\rangle)$, the singular locus $\mathrm{Sing}(\tilde{\omega})$ of any lifting $\tilde{\omega} = bdX - adY \in \Omega^1_{k[X,Y]/k}$ of a torsion Kähler differential form $\omega \in \Omega^1_{\mathscr{V}/k}(\mathscr{V})$ is, as usually, the subset of $\mathscr{V}$ formed by the maximal ideals $\mathfrak{m}$ of $k[X,Y]$, containing f, a and b. We observe that, if we restrict this locus to the singular points of $\mathscr{V}$, this restriction does not depend on the choice of the lifting. Then, we set $\mathrm{Sing}(\omega) := \mathrm{Sing}(\tilde{\omega}) \cap \mathrm{Sing}(\mathscr{V})$ and call it the *singular locus* of ω. (See subsection 3.4 for a more intrinsic definition.) By subsection 1.2, we know that the associated k-derivation $D_\omega = a\partial_X + b\partial_Y$ induces a k-derivation on $\mathscr{V}$ and, by a simple observation (e.g., see [13, Lemma 4]), we conclude that $\mathrm{Sing}(\mathscr{V}) \subset \mathrm{Sing}(\omega)$. Then the second problem of this article can be formulated as follows:

Question 1.6. *Let k be a field of characteristic zero. Let $\mathscr{V}$ be an integral affine k-variety. How do the singular locus of $\mathscr{V}$ compare with the locus where ω vanishes ?*

To the best of our knowledge, Question 1.6 is open. In section 8, we provide a proof of the following positive element of answer to this question:

Theorem 1.7. *Let k be a field of characteristic zero. Let $N \geq 1$ be an integer. Let $\mathscr{V}$ be an integral hypersurface of $\mathbf{A}^N_k$. Let $\omega \in \mathrm{Tors}(\Omega^1_{\mathscr{V}/k}(\mathscr{V}))$ be a torsion Kähler differential form. Then, for every lifting $\tilde{\omega}$ in $\Omega^1_{\mathbf{A}^N_k/k}(\mathbf{A}^N_k)$, for every non-cylindrical closed point $x \in \mathscr{V}$, we have $x \in \mathrm{Sing}(\tilde{\omega})$.*

In section 5, we introduce the terminology of cylindrical points. In particular, a non-cylindrical closed point of local dimension at least one of a k-variety necessarily is a singular point. Roughly speaking non-cylindrical singularities form a large natural class of singularities suggested for example by the classical result [13, Lemma 4]. This class corresponds to singularities which appear in the highest dimension and, in this way, can be interpreted as *wild* singularities in contrast to cylindrical singularities, the study of which being reduced, by the very definition, to smaller dimensional singularities. Isolated singularities are

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