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Desingularization of regular algebras

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A B S T R A C T

We identify families of commutative rings that can be written as a direct limit of a directed system of noetherian regular rings and investigate the homological properties of such rings.

1. Introduction

The goal of this work is to identify rings R that can be realized as a direct limit of a directed system $\{R_i : i \in \Gamma\}$ of noetherian regular rings (which we then call a *desingularization* of R), and to investigate the homological properties of such an R. We emphasize that the poset Γ is filtered. A paradigm for this, and one of the motivation for this work, is a result of Zariski [29] (and Popescu [25]):

Theorem 1.1 *(Zariski–Popescu). Let $(V, \mathfrak{m})$ be a valuation domain containing a field k of zero characteristic. Then V has a desingularization.*

It may be interesting to mention that the construction of desingularizations goes back to Akizuki [1] and Nagata [20]. Recall from [6] that a ring is said to be *regular*, if each finitely generated ideal has finite projective dimension. A ring is called *coherent*, if its finitely generated ideals are finitely presented. Our first result in Section 2 is:

Proposition 1.2. *Let R be a ring that has a desingularization and $\mathfrak{p}$ a finitely generated prime ideal in R. If $R_{\mathfrak{p}}$ is coherent, then $R_{\mathfrak{p}}$ is regular.*

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Also, Section 2 is devoted to computing the homological dimensions of an ideal I of a ring with a desingularization $\{R_i : i \in \Gamma\}$. We do this by imposing some additional assumptions both on the ideal I, the rings R_i and the poset Γ. This may extend some results by Berstein [7] and Osofsky [24].

A *quasilocal* ring is a ring with a unique maximal ideal. A local ring is a noetherian quasilocal ring. There are many definitions for the regularity condition in non-noetherian rings (see e.g. [16]). One of these is the notion of *super regularity*. This notion was first introduced by Vasconcelos [28]. A coherent quasilocal ring is called super regular if its global dimension is finite and equal to its weak dimension. Section 3 deals with a desingularization of super regular rings. Our first result in this direction is Proposition 3.3:

Proposition 1.3. *Let $\{(R_i, \mathfrak{m}_i)\}$ be a directed system of local rings with the property that $\mathfrak{m}_i^2 = \mathfrak{m}_i \cap \mathfrak{m}_{i+1}^2$. If $R := \varinjlim R_i$ is coherent and super regular, then each R_i is regular.*

We present an application of the notion of super regularity: we compute the global dimension of certain *perfect* algebras. To this end, suppose R is a complete local domain which is not a field and suppose that its *perfect closure* R^∞ is coherent. In Proposition 3.4 we show that

$$\operatorname{gl.dim}(R^\infty) = \dim R + 1.$$

Let $\{R_i\}$ be a pure directed system of local rings and suppose that the maximal ideal of $R := \varinjlim R_i$ has a finite free resolution. In Proposition 4.1, we show R is noetherian and regular. Let F be a finite field. In Proposition 5.3 we present a desingularization of $\prod_{\mathbb{N}} F$. This has some applications. For example, $\prod_{\mathbb{N}} F$ is stably coherent.

We cite [14] as a reference book on commutative coherent rings.

2. Homological properties of a desingularization

We start by introducing some notation. By $\operatorname{p.dim}_R(-)$ (resp. $\operatorname{fl.dim}_R(-)$), we mean projective dimension (resp. flat dimension) of an R-module. Denote the ith Koszul homology module of R with respect to $\underline{x} := x_1, \ldots, x_n$ by $H_i(\underline{x}; R)$.

Remark 2.1. Let $\{R_i : i \in \Gamma\}$ be a directed system of rings and let $\underline{x} := x_1, \ldots, x_n$ be in $R := \varinjlim R_i$. Let i_0 be such that $\underline{x} \subset R_{i_0}$. Then $\varinjlim_{i \geq i_0} H_\bullet(\underline{x}, R_i) \simeq H_\bullet(\underline{x}, R)$.

Proof. This is straightforward and we leave it to the reader. □

Definition 2.2. Let $(R, \mathfrak{m})$ be a quasilocal ring. Suppose $\mathfrak{m}$ is generated by a finite sequence of elements $\underline{x} := x_1, \ldots, x_n$. Recall from Kabele [16] that R is H_1*-regular*, if $H_1(\underline{x}, R) = 0$. Also, R is called *Koszul regular*, if $H_i(\underline{x}, R) = 0$ for all $i > 0$.

In general, H_1-regular rings are not Koszul regular, see [16].

Lemma 2.3. *Let $(R, \mathfrak{m})$ be a coherent H_1-regular ring. Then R is Koszul regular.*

Proof. Coherence regular local rings are integral domains. Recall from Definition 2.2 that $\mathfrak{m}$ is finitely generated. Let $\underline{x} := x_1, \ldots, x_n$ be a generating set for $\mathfrak{m}$. Since R is coherent and in view of [3, Lemma 3.7], the R-module $H_i(\underline{y}, R)$ is finitely generated, where $\underline{y}$ is a finite sequence of elements. By using basic properties of Koszul homologies and by an easy induction one may show that $H_i(x_1, \ldots, x_j; R) = 0$ for all $i > 0$ and all j. We left the routine details to the reader, please see [19, pp. 127–128]. □