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On the topology of valuation-theoretic representations of integrally closed domains

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ABSTRACT

Let F be a field. For each nonempty subset X of the Zariski–Riemann space of valuation rings of F, let $A(X) = \bigcap_{V \in X} V$ and $J(X) = \bigcap_{V \in X} \mathfrak{M}_V$, where $\mathfrak{M}_V$ denotes the maximal ideal of V. We examine connections between topological features of X and the algebraic structure of the ring $A(X)$. We show that if $J(X) \neq 0$ and $A(X)$ is a completely integrally closed local ring that is not a valuation ring of F, then there is a space Y of valuation rings of F that is perfect in the patch topology such that $A(X) = A(Y)$. If any countable subset of points is removed from Y, then the resulting set remains a representation of $A(X)$. Additionally, if F is a countable field, the set Y can be chosen homeomorphic to the Cantor set. We apply these results to study properties of the ring $A(X)$ with specific focus on topological conditions that guarantee $A(X)$ is a Prüfer domain, a feature that is reflected in the Zariski–Riemann space when viewed as a locally ringed space. We also classify the rings $A(X)$ where X has finitely many patch limit points, thus giving a topological generalization of the class of Krull domains, one that includes interesting Prüfer domains. To illustrate the latter, we show how an intersection of valuation rings arising naturally in the study of local quadratic transformations of a regular local ring can be described using these techniques.

1. Introduction

By a classical theorem of Krull, every integrally closed subring A of a field F is an intersection of valuation rings of F [21, Theorem 10.4, p. 73]. If also A is local but not a valuation ring of F, then A is an intersection of infinitely many valuation rings [24, (11.11), p. 38]. If in addition A is completely integrally closed, then any set of valuation rings of F that dominate A and whose intersection is A must be uncountable. (See Corollary 3.7.) Rather than the cardinality of representing sets of valuation rings of F, our focus on this article is on qualitative features of these sets. We describe these features topologically as part of a goal of connecting topological properties of subspaces X of the Zariski–Riemann space $\mathrm{Zar}(F)$ of F with algebraic

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features of the intersection ring $A = \bigcap_{V \in X} V$. We work mostly in the patch topology on Zar(F), with special emphasis on the collection lim(X) of patch limit points of the subset X of Zar(F).

For each nonempty subset X of Zar(F), let $A(X) = \bigcap_{V \in X} V$ and $J(X) = \bigcap_{V \in X} \mathfrak{M}_V$, where $\mathfrak{M}_V$ denotes the maximal ideal of V. We prove in Lemma 3.2 that if X is infinite and $A(X)$ is a local ring, then $J(\lim X) \subseteq A(X)$. While it is clear that $J(\lim X)$ is an ideal of $A(\lim X)$, the content of the lemma is that this radical ideal is also an ideal of the subring $A(X)$ of $A(\lim X)$. We use this fact to show in Theorem 3.5 that if $J(X) \neq 0$ and A is completely integrally closed and local but not a valuation ring of F, then $A(X) = A(\lim X)$. In fact, it is shown that lim(X) can be replaced with a perfect space, that is, a space Y for with $Y = \lim(Y)$. Along with the Baire Category Theorem, this implies in Corollary 3.7 that any countable subset of lim(X) can be removed and the resulting set remains a representation of A. Thus no completely integrally closed local subring of F that is not a valuation ring of F can be written as a countable intersection of valuation rings that dominate the ring. Additionally, if F is a countable field, we conclude in Corollary 3.6 that lim(X) contains a subset Y such that $A(X) = A(Y)$ and Y is homeomorphic to the Cantor set. In the case of a countable field F, every local completely integrally closed subring of F that is not a valuation ring of F has a representation that is homeomorphic in the patch topology to the Cantor set. (By a *representation* of an integrally closed subring A of F we mean a subset Y of Zar(F) such that $A = A(Y)$.)

In Section 4 we apply the results of Section 3 to develop sufficient topological conditions for an intersection of valuation rings to be a Prüfer domain; that is, a domain A for which each localization of A at a maximal ideal is a valuation domain. In general the question of whether an intersection of valuation rings is a Prüfer domain is very subtle; see [11,20,26,28,29,32,33] and their references for various approaches to this question. While Prüfer domains have been thoroughly investigated in Multiplicative Ideal Theory (see for example [8,10,19]), our interest here lies more in the point of view that these rings form the "coordinate rings" of sets X in Zar(F) that are non-degenerate in the sense that every localization of $A(X)$ lies in X. This is expressed geometrically as asserting that X lies in an affine scheme of Zar(F), where Zar(F) is viewed as a locally ringed space. We discuss this viewpoint in more detail at the beginning of Section 4. We prove for example in Corollary 4.4 that if X is a set of rank one valuation rings of F such that $J(X) \neq 0$ and $A(\lim X)$ is a Prüfer domain for which the intersection of nonzero prime ideals is nonzero, then $A(X)$ is a Prüfer domain. In Sections 4 and 5 we prove several other results in this same spirit, where the emphasis is on determination of algebraic properties of $A(X)$ from algebraic properties of the generally larger ring $A(\lim X)$. This continues the theme in Section 3 of using the patch limit points of X to shed light on $A(X)$.

In Section 5 we focus on conditions that limit the size of $\lim X$. For example, in Corollary 5.4 we classify the rings $A(X)$ where X is a set of rank one valuation rings in Zar(F) with finitely many patch limit points. Since such rings include the class of Krull domains that have quotient field F, this setting can be viewed as a topological generalization of Krull domains. But the class also includes interesting Prüfer domains that behave very differently than Krull domains. To illustrate the latter assertion, we show that a subspace X of Zar(F) associated to a sequence of iterated quadratic transforms of a regular local ring yields a Prüfer domain such as those studied in Section 4. It follows that the boundary valuation ring of a sequence of iterated quadratic transforms of regular local rings is a localization of the order valuation rings associated to each of the quadratic transforms.

For some related examples of recent work on the topology of the Zariski–Riemann space of a field, see [7,9,26–29] and their references. Our approach is related to that of [29] where it is shown that if X is a quasicompact set of rank one valuation rings in Zar(F) with $J(X) \neq 0$, then $A(X)$ is a one-dimensional Prüfer domain with quotient field F and nonzero Jacobson radical. Similarly, in the present article we see that the topological condition of having finitely many patch limit points suffices to determine algebraic features of the intersection ring $A(X)$. In both the present article and [29] Prüfer domains play a key role because these correspond to affine schemes in Zar(F) when the Zariski–Riemann space is viewed as a locally ringed space; see [28] for more on this point of view. In general, the determination of which subsets of

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