ARTICLE IN PRESS JPAA:5757

Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

# Relative singularity categories, Gorenstein objects and silting theory ☆

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ARTICLE INFO

*Article history:*
Received 28 October 2015
Received in revised form 11 August 2017
Available online xxxx
Communicated by S. Koenig

*MSC:*
Primary: 18E30; 16E05; secondary: 18G35; 16G10

ABSTRACT

We study singularity categories through Gorenstein objects in triangulated categories and silting theory. Let $\omega$ be a presilting subcategory of a triangulated category $\mathcal{T}$. We introduce the notion of $\omega$-Gorenstein objects. It is a far extended version in triangulated categories of Gorenstein projective modules and Gorenstein injective modules. We prove that the stable category $\underline{\mathcal{G}_\omega} := \mathcal{G}_\omega/\mathrm{add}\omega$ of $\mathcal{G}_\omega$ modulo add$\omega$, where $\mathcal{G}_\omega$ is the subcategory of all $\omega$-Gorenstein objects, is a triangulated category. Moreover, we prove that, under some conditions, the triangulated category $\underline{\mathcal{G}_\omega}$ is triangle equivalent to the relative singularity category of $\mathcal{T}$ with respect to the thick subcategory generated by $\omega$. As applications, we obtain the following characterizations of singularity categories which partially extend classical results (usually in the context of Gorenstein rings) to some more general settings. (1) For a ring $R$ of finite Gorenstein global dimension, there are triangle equivalences between $\underline{\mathcal{G}P}$ (the stable category of Gorenstein projective modules), $\underline{\mathcal{G}I}$ (the stable category of Gorenstein injective modules), and $\underline{\mathcal{G}_{\mathrm{Add}M}}$ (where $M$ is any big silting complex in $\mathcal{D}^b(\mathrm{Mod}R)$) and the big singularity category $\mathcal{D}_{\mathrm{Sg}}(R)$. (2) For a left coherent ring $R$ of finite Gorenstein global dimension, there are triangle equivalences between $\underline{\mathcal{G}p}$ (the stable category of finitely generated Gorenstein projective left modules), and $\underline{\mathcal{G}_{\mathrm{add}M}}$ (where $M$ is any silting complex in $\mathcal{D}^b(R)$) and the singularity category $\mathcal{D}_{sg}(R)$.



## 1. Introduction

Singularity categories (also called stable derived categories [21] at the compact level) are important in the study of varieties and rings of infinite global dimension. For an algebraic variety $X$, the singularity category is given by the Verdier's quotient triangulated category $\mathcal{D}_{sg}(X) := \mathcal{D}^b(\mathrm{coh}X)/\mathrm{perf}X$, where $\mathcal{D}^b(\mathrm{coh}X)$ is the bounded derived category of coherent sheaves on $X$, and perf$X$ is its full subcategory of perfect complexes

☆ Supported by the National Science Foundation of China (Grant No. 11771212) and a project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

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https://doi.org/10.1016/j.jpaa.2017.09.014
0022-4049/

Please cite this article in press as: J. Wei, Relative singularity categories, Gorenstein objects and silting theory, J. Pure Appl. Algebra (2017), https://doi.org/10.1016/j.jpaa.2017.09.014

[23]. $\mathcal{D}_{sg}(X) = 0$ if and only if $X$ is smooth. In representation theory of algebras, singularity categories are defined as the Verdier's quotient triangulated categories $\mathcal{D}_{sg}(R) := \mathcal{D}^b(R)/\mathcal{K}^b(\mathcal{P})$, where $\mathcal{D}^b(R)$ is the bounded derived category of finitely presented modules over a (left) coherent ring $R$, and $\mathcal{K}^b(\mathcal{P})$ is the bounded homotopy category of finitely generated projective $R$-modules [10,11]. The singularity category $\mathcal{D}_{sg}(R)$ is 0 if and only if any finitely presented (left) module has finite projective dimension.

Similar quotient triangulated categories were also studied by several people, such as $\mathcal{D}_{\mathrm{Sg}}(R) := \mathcal{D}^b(\mathrm{Mod}R)/\mathcal{K}^b(\mathrm{Proj})$ [7], $\mathcal{D}^b(\mathrm{Mod}R)/\mathcal{K}^b(\mathrm{Inj})$ [7], $\mathcal{D}^b(R)/\mathcal{K}^b(\mathcal{I})$ [15], $\mathcal{D}^b(R)/\mathcal{K}^b(\mathrm{add}T)$ [12], where Mod$R$ (Proj, Inj, $\mathcal{I}$, resp.) denotes the category of all (projective, injective, finitely generated injective, resp.) $R$-modules, over a ring $R$. In [11], a more general notion of relative singularity categories $\mathcal{D}_\omega(\mathcal{A}) := \mathcal{D}^b(\mathcal{A})/\mathcal{K}^b(\omega)$ was introduced, for $\mathcal{A}$ an abelian category and $\omega$ a selforthogonal additive subcategory of $\mathcal{A}$. The notion unifies the previous various singularity categories.

Gorenstein projective modules (or Cohen–Macaulay objects) and Gorenstein injective modules over Gorenstein rings play important roles in the study of the above relative singularity categories [7,10,11, 15], etc. This theory goes back to Auslander's notion of modules of $G$-dimension zero [4]. Later it was extensively studied and developed by Avramov and Foxby [6], Buchweitz [10], Enochs and Jenda [13], etc. in various ways. The Gorenstein global dimension of a ring is well-defined. It coincides with the supremum of all Gorenstein projective dimensions of modules as well as the supremum of all Gorenstein injective dimensions of modules [7,9]. An important class of rings with finite Gorenstein global dimension are Gorenstein rings, that is, left and right noetherian rings with finite left and right self-injective dimensions. However, there are also non-noetherian rings with finite Gorenstein global dimension [22].

In this paper, we aim to study singularity categories through silting theory. Recall that silting theory was first introduced by Keller and Vossieck [18]. It was re-interpreted by Aihara and Iyama [1] from the point of view of mutations. In [28], the author studied so called semi-tilting complexes from the point of view of comparison between tilting complexes [20,25] and tilting modules. However, it turns out that semi-tilting complexes are just silting complexes.

The paper is organized as follows. After the introduction, we present in Section 2 a new extension of Gorenstein objects in triangulated categories (compare [2]). Then we introduce the relative singularity category $\mathcal{T}_\omega$ for a triangulated category $\mathcal{T}$ with a presilting subcategory $\omega$ to be the Verdier quotient $\mathcal{T}/\mathrm{add}\omega$, by employing methods and techniques from silting theory in triangulated categories [28]. This generalizes the above-mentioned variants of singularity categories.

The following is our main result in Section 2. It characterizes the relative singularity category $\mathcal{T}_\omega$ as the stable category of $\omega$-Gorenstein objects in $\mathcal{T}$, under some conditions (see Theorem 2.6 for details) on certain subcategories ${}_\omega\mathcal{X}$ and $\mathcal{X}_\omega$ defined in the second paragraph before Lemma 2.2.

**Theorem A.** *Let $\mathcal{T}$ be a triangulated category and $\omega$ be a presilting subcategory of $\mathcal{T}$. If $\mathcal{T} = \langle {}_\omega\mathcal{X} \rangle = \langle \mathcal{X}_\omega \rangle$, then there is a triangle equivalence between $\underline{\mathcal{G}_\omega}$ and the relative singularity category $\mathcal{T}_\omega$.*

Applications of Theorem A are given in Section 3, where we obtain the following results which extend classical results [7,10,11], etc. to more general settings (Theorem B (2)). They also provide characterizations of singularity categories in terms of silting complexes, see Section 3 for details.

**Theorem B.**

(1) *Let $R$ be a ring with finite Gorenstein global dimension. Then there are triangle equivalences between the following triangulated categories.*
   (i) $\underline{\mathcal{GP}}$ *(the stable category of Gorenstein projective modules),*
   (ii) $\underline{\mathcal{GI}}$ *(the stable category of Gorenstein injective modules),*

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