ARTICLE IN PRESS JPAA:5714

Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Homogeneous nilradicals over semigroup graded rings

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ARTICLE INFO

Article history:
Received 12 January 2017
Received in revised form 2 June 2017
Available online xxxx
Communicated by S. Iyengar

MSC:
Primary: 16N40; 16W50; secondary: 16N20; 16N60; 16N80; 20M10; 06F05

ABSTRACT

In this paper we study the homogeneity of radicals defined by nilpotence or primality conditions, in rings graded by a semigroup S. When S is a unique product semigroup, we show that the right (and left) strongly prime and uniformly strongly prime radicals are homogeneous, and an even stronger result holds for the generalized nilradical. We further prove that rings graded by torsion-free, nilpotent groups have homogeneous upper nilradical. We conclude by showing that non-semiprime rings graded by a large class of semigroups must always contain nonzero homogeneous nilpotent ideals.

1. Introduction

To understand the structure of a ring it is extremely useful to consider many different radicals of the ring. A natural problem in ring theory is to determine which radicals of a graded ring are generated by homogeneous elements, and moreover describe the elements of such radicals under (graded) extensions. For instance, the famous (and currently open) Köthe Conjecture posits that any nil one-sided ideal is contained in a nil two-sided ideal. Surprisingly, this is equivalent to showing $R[x]$ is Jacobson radical if (and only if) R is a nil ring.

Many homogeneity results for classical radicals are known, with arguably one of the earliest and most influential results being that of Bergman [1], who proved that the Jacobson radical of any $\mathbb{Z}$-graded ring is homogeneous. More generally, it has been proven in [10] that if G is a group which is residually p-finite for two distinct primes p and R is a G-graded ring, then $J(R)$ is homogeneous. Since this large class of groups

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http://dx.doi.org/10.1016/j.jpaa.2017.07.009
0022-4049/

contains all free groups and finitely generated torsion-free nilpotent groups, any ring graded by such a group must have a homogeneous Jacobson radical.

In this paper we will work more broadly with semigroup graded rings (see Kelarev's book [14] as a resource for information on graded rings), and a very natural class of semigroups in this context are the so-called *unique product semigroups*, or u.p.-semigroups for short. We will recall the definition and basic properties of such semigroups in Section 2, and give an interesting generalization of the unique product property a little later in Section 3. By two well-known theorems of Jespers, Krempa, and Puczylowski found in [9], if S is a u.p.-semigroup, then any S-graded ring has homogeneous prime radical and Levitzki (or locally nilpotent) radical. It is still open whether this result is true for the Jacobson radical and upper nilradical, but some partial results are known (see [8] and [11] for a collection of such results). However, there are two known necessary conditions; the semigroup must be cancellative and torsion free (see [18]).

For commutative semigroups, the situation is more understood. The following proposition summarizes information that can be found in [9] and [18].

Proposition A. *Let S be a commutative semigroup. The prime radical (respectively, Levitzki radical, Jacobson radical, upper nilradical, bounded nilradical, Wedderburn radical) is homogeneous in every S-graded ring if and only if S is cancellative and torsion-free.*

This proposition may give the impression that cancellativity and torsion-freeness are always necessary (at least when considering "nice" radicals). Indeed, there are ranges of radicals $\mathfrak{F}$ where these two properties hold in any semigroup S such that $\mathfrak{F}(R)$ is homogeneous whenever R is S-graded (see [18, Theorem 6 and Proposition 11]). However, we will see that for those radicals which fall outside these ranges, those properties are not necessary.

In Section 3 we work with the right strongly prime radical $s_r(R)$, the uniformly strongly prime radical $u(R)$, and the generalized nilradical $N_g(R)$ for a ring R. (The definitions and basic properties of these radicals are recalled in that section, but the reader may consult [4] for additional information. In particular, there is a nice diagram showing the relationship between these radicals and those discussed above, on page 293 of [4]. We will cite the relevant proofs when we need to use any of these containments.) The first main result of this paper is the following:

Theorem B. *If S is a u.p.-semigroup and R is an S-graded ring, then $s_r(R)$, $u(R)$, and $N_g(R)$ are homogeneous.*

Theorem B incorporates information from Theorems 3.7, 3.10, and 3.12 below. In fact, we obtain a slight improvement on the result for the generalized nilradical. Moreover, as a corollary, we find that we can add $s_r(R)$ and $u(R)$ to the list of radicals given in Proposition A.

We next turn to the context of *group* (rather than semigroup) graded rings. In Section 4 we study the upper nilradical of group graded rings. Much less is known regarding the homogeneity of the upper nilradical of such a graded ring in comparison with the other standard radicals. However, by bootstrapping an argument due to Smoktunowicz from [26], we are able to prove the second main result of the paper, which will appear as Theorem 4.1, and which we restate here.

Theorem C. *If G is a torsion-free, nilpotent group and R is a G-graded ring, then $\mathrm{Nil}^*(R)$ is homogeneous.*

As a consequence of this theorem, we are able to generalize [18, Theorem 13] for group graded rings.

We end the paper in Section 5 by also proving homogeneity results for the Wedderburn radical of a ring. Our final main result, appearing as Theorem 5.2 is the following: