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Criteria for a ring to have a left Noetherian left quotient ring

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ABSTRACT

Two criteria are given for a ring to have a left Noetherian left quotient ring (to find a criterion was an open problem since 70's). It is proved that each such ring has only *finitely many* maximal left denominator sets.

1. Introduction

In this paper, module means a left module, and the following notation is fixed:

- R is a ring with 1;
- $\mathcal{C} = \mathcal{C}_R$ is the set of *regular* elements of the ring R (i.e., $\mathcal{C}$ is the set of non-zero-divisors of the ring R);
- $Q = Q(R) := Q_{l,cl}(R) := \mathcal{C}^{-1}R$ is the *left quotient ring* (the *classical left ring of fractions*) of the ring R (if it exists) and Q^* is the group of units of Q;
- $\mathfrak{n} = \mathfrak{n}_R$ is the prime radical of R, $\nu \in \mathbb{N} \cup \{\infty\}$ is its *nilpotency degree* ($\mathfrak{n}^\nu \neq 0$ but $\mathfrak{n}^{\nu+1} = 0$) and $\mathcal{N}_i := \mathfrak{n}^i/\mathfrak{n}^{i+1}$ for $i \in \mathbb{N}$;
- $\overline{R} := R/\mathfrak{n}$ and $\pi : R \to \overline{R}$, $r \mapsto \overline{r} = r + \mathfrak{n}$;
- $\overline{\mathcal{C}} := \mathcal{C}_{\overline{R}}$ is the set of regular elements of the ring $\overline{R}$ and $\overline{Q} := \overline{\mathcal{C}}^{-1}\overline{R}$ is its left quotient ring;
- $\widetilde{\mathcal{C}} := \pi(\mathcal{C})$, $\widetilde{Q} := \widetilde{\mathcal{C}}^{-1}\overline{R}$ and $\mathcal{C}^\dagger := \mathcal{C}_{\widetilde{Q}}$ is the set of regular elements of the ring $\widetilde{Q}$;
- $S_l = S_l(R)$ is the *largest left Ore* of R that consists of regular elements and $Q_l = Q_l(R) := S_l(R)^{-1}R$ is the *largest left quotient ring* of R [5, Theorem 2.1];
- $\mathrm{Ore}_l(R) := \{S \,|\, S \text{ is a left Ore set in } R\}$;
- $\mathrm{Den}_l(R) := \{S \,|\, S \text{ is a left denominator set in } R\}$.

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A question of describing rings such that their (left) quotient ring satisfies certain conditions tends to be a challenging one. A first step was done by Goldie (1960) [12] who gave a criterion for a ring R to have a semisimple Artinian quotient ring $Q(R)$. The case when $Q(R)$ is a simple Artinian ring is due to Goldie (1960) [10] and Lesieur and Croisot (1958) [15]. Criteria for a ring to have a left Artinian left quotient ring were given by Small (1966) [20,21]; Robson (1967) [19], Tachikawa (1971) [25], Hajarnavis (1972) [14], Warfield (1981) [26] and the author (2013) [2].

Remark. In the paper, a statement that 'If R ... then $Q(R)$ ...' means that 'If R satisfies ... then $Q(R)$ **exists** and satisfies ...'.

Criteria for a ring to have a left Noetherian left quotient ring. The aim of the paper is to give two criteria for a ring R to have a left Noetherian left quotient ring (Theorem 1.2 and Theorem 1.3). The case when R is a *semiprime* ring is a very easy special case.

Theorem 1.1. *Let R be a semiprime ring. Then the following statements are equivalent.*

1. *$Q(R)$ is a left Noetherian ring.*
2. *$Q(R)$ is a semisimple ring.*
3. *R is a semiprime left Goldie ring.*
4. *$Q_l(R)$ is a left Noetherian ring.*
5. *$Q_l(R)$ is a semisimple ring.*

If one of the equivalent conditions holds then $S_l(R) = \mathcal{C}_R$ and $Q(R) = Q_l(R)$. In particular, if the left quotient ring $Q(R)$ (respectively, $Q_l(R)$) is not a semisimple ring then the ring $Q(R)$ (respectively, $Q_l(R)$) is not left Noetherian.

The proof of Theorem 1.1 is given in Section 3.

Example. ([1].) The ring $\mathbb{I}_1 := K\langle x, \frac{d}{dx}, \int\rangle$ of polynomial integro-differential operators over a field K of characteristic zero is a semiprime ring but not left Goldie (as it contains infinite direct sums of non-zero left ideals). Therefore, the largest left quotient ring $Q_l(\mathbb{I}_1)$ is not a left Noetherian ring (moreover, the left quotient ring $Q(\mathbb{I}_1)$ does not exists). The ring $Q_l(\mathbb{I}_1)$ and the largest regular left Ore set $S_l(\mathbb{I}_1)$ of $\mathbb{I}_1$ were described explicitly in [1].

The first criterion for a ring to have a left Noetherian left quotient ring is below, its proof is given in Section 3.

Theorem 1.2. *Let R be a ring. The following statements are equivalent.*

1. *The ring R has a left Noetherian left quotient ring $Q(R)$.*
2. (a) $\widetilde{\mathcal{C}} \subseteq \overline{\mathcal{C}}$.
   (b) $\widetilde{\mathcal{C}} \in \mathrm{Ore}_l(\overline{R})$.
   (c) $\widetilde{Q} = \widetilde{\mathcal{C}}^{-1}\overline{R}$ *is a left Noetherian ring.*
   (d) $\mathfrak{n}$ *is a nilpotent ideal of the ring R.*
   (e) *The $\widetilde{Q}$-modules $\widetilde{\mathcal{C}}^{-1}\mathcal{N}_i$, $i = 1, \ldots, \nu$, are finitely generated (where ν is the nilpotency degree of $\mathfrak{n}$ and $\mathcal{N}_i := \mathfrak{n}^i/\mathfrak{n}^{i+1}$).*
   (f) *For each element $\overline{c} \in \widetilde{\mathcal{C}}$, the left $\overline{R}$-module $\mathcal{N}_i/\mathcal{N}_i\overline{c}$ is $\widetilde{\mathcal{C}}$-torsion for $i = 1, \ldots, \nu$.*

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