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Scalar extensions of categorical resolutions of singularities

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A B S T R A C T

Let X be a quasi-compact, separated scheme over a field k and we can consider the categorical resolution of singularities of X. In this paper let k'/k be a field extension and we study the scalar extension of a categorical resolution of singularities of X and we show how it gives a categorical resolution of the base change scheme $X_{k'}$. Our construction involves the scalar extension of derived categories of DG-modules over a DG algebra. As an application we use the technique of scalar extension developed in this paper to prove the non-existence of full exceptional collections of categorical resolutions for a projective curve of genus ≥ 1 over a non-algebraically closed field.

1. Introduction

For a scheme X over a field k, the derived category $\mathrm{D}(X)$ of quasi-coherent $\mathcal{O}_X$-modules plays an important role in the study of the geometry of X. In particular, a *categorical resolution* of singularities of X is defined to be a smooth triangulated category $\mathscr{T}$ together with an adjoint pair $\pi^* : \mathrm{D}(X) \rightleftarrows \mathscr{T} : \pi_*$ which satisfies certain properties. See [7] or Definition 3.1 below for details.

On the other hand, base change techniques are also ubiquitous in algebraic geometry. In [13], a theory of scalar extensions of triangulated categories has been developed and applied to derived categories of varieties.

In this paper we define and study the scalar extension of categorical resolutions. The difficulty is to find the scalar extensions of the adjoint pair (π^*, π_*). To solve this problem we modify the definition of categorical resolution: inspired by [8], we define an *algebraic categorical resolution* of X to be a triple (A, B, T) where A is a differential graded (DG) algebra such that $\mathrm{D}(X) \simeq \mathrm{D}(A)$, B is a smooth DG algebra and T is an A–B bimodule which satisfies certain properties. See Definition 3.4 below for more details. In some important cases, which include the cases we are most interested in, these two definitions are equivalent. For the comparison of different definitions of categorical resolution see Proposition 3.4 below.

The advantage of algebraic categorical resolution is that it is compatible with base field extensions. One of the main results in this paper is the following proposition.

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Proposition 1.1 *(See Proposition 4.11 below). Let X be a projective variety over a field k. If (A, B, T) is an algebraic categorical resolution of X, then $(A_{k'}, B_{k'}, T_{k'})$ is an algebraic categorical resolution of the base change variety $X_{k'}$.*

As an application we study the categorical resolution of projective curves X over a non-algebraically closed field k. Using the technique of scalar extension we obtain the following theorems which generalize the main results in [17].

Theorem 1.2 *(See Theorem 5.4 below). Let X be a projective curve over a field k. Then X has a categorical resolution which admits a full exceptional collection if and only if the geometric genus of X is 0.*

Theorem 1.3 *(See Theorem 5.5 below). Let X be a projective curve with geometric genus ≥ 1 over a field k and $(\mathscr{T}, \pi^*, \pi_*)$ be a categorical resolution of X. Then $\mathscr{T}^c$ cannot have a tilting object, moreover there cannot be a finite dimensional k-algebra Λ of finite global dimension such that*

$$\mathscr{T}^c \simeq \mathrm{D}^b(\Lambda\text{-}mod).$$

This paper is organized as follows: In Section 2 we quickly review triangulated categories, DG categories and DG algebras. In Section 3 we review and compare different definitions of categorical resolutions. In Section 4.1 we study the scalar extension of derived categories of DG algebras and in Section 4.2 we study the scalar extension of categorical resolutions. In Section 5 we use the technique of scalar extension to study categorical resolutions of projective curves over a non-algebraically closed field.

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2. Preliminaries

2.1. Review of some concepts on triangulated categories

Let $\mathcal{T}$ be a triangulated category. $\mathcal{T}$ is called *cocomplete* if it has arbitrary direct sums. An object E of a cocomplete triangulated category $\mathcal{T}$ is called *compact* if the functor $\mathrm{Hom}_{\mathcal{T}}(E, -)$ preserves arbitrary direct sums. Let $\mathcal{T}^c$ denote the full triangulated subcategory of $\mathcal{T}$ consisting of compact objects.

Let $\mathcal{I}$ be a set of objects of $\mathcal{T}$. We say $\mathcal{I}$ *generates* $\mathcal{T}$ if for any object N of $\mathcal{T}$, $\mathrm{Hom}_{\mathcal{T}}(E, N[i]) = 0$ for any $E \in \mathcal{I}$ and $i \in \mathbb{Z}$ implies $N = 0$. We say a cocomplete triangulated category $\mathcal{T}$ is *compactly generated* if it is generated by a set of compact objects. An object E of $\mathcal{T}$ is called a generator of $\mathcal{T}$ if the set $\{E\}$ generates $\mathcal{T}$.

We have the following well-known result.

Lemma 2.1. *Let $\mathcal{T}$ be a cocomplete triangulated category. If a set of objects $\mathcal{E} \subset \mathcal{T}^c$ generates $\mathcal{T}$, then $\mathcal{T}$ coincides with the smallest strictly full triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{E}$ and is closed under direct sums. Recall a subcategory is strictly full if it is full and closed under isomorphism.*

Proof. See the proof of [12] Theorem 4.22. □

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