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Levelwise modules over separable monads on stable derivators

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ABSTRACT

We show that given a separable cocontinuous monad on a triangulated derivator, the levelwise Eilenberg–Moore categories of modules glue together to a triangulated derivator. As an application, we give examples of derivators that are stable but not strong.

0. Introduction

Given a monoid object A in a tensor triangulated category, we can consider the category of modules over it. This category is rarely useful from a homotopical perspective since it often fails to inherit a natural triangulated structure (see Example 7.14). A notable exception is when the monoid A is separable (see [1]). Such monoids appear frequently in practice: commutative étale algebras in commutative algebra [1, Corollary 6.6], étale extensions in algebraic geometry [3, Theorem 3.5 and Remark 3.8], $k(G/H)$ for subgroups $H < G$ of finite index in representation theory [2, Theorem 1.2], $\Sigma^\infty(G/H)_+$ for H a closed subgroup of finite index of a compact Lie group in equivariant stable homotopy theory [4, Theorem 1.1], and in other equivariant settings [4, Theorems 1.2, 1.3]. More generally, in [1] the author proves that given a separable exact monad on a triangulated category $\mathscr{C}$, the category of modules over that monad inherits a triangulation from that of $\mathscr{C}$ in a compatible way (this includes Bousfield localization as a special case). Our purpose is to prove a similar statement for triangulated derivators:

Main Theorem. *Let $M : \mathbb{D} \to \mathbb{D}$ a cocontinuous separable monad on a triangulated idempotent-complete derivator $\mathbb{D}$. Then the levelwise Eilenberg–Moore categories of M define a triangulated derivator M-$\mathbf{Mod}_{\mathbb{D}}$.*

In particular, our theorem gives a new proof that modules over a separable monoid in a tensor triangulated category $\mathscr{C}$ inherit a natural triangulation from that of $\mathscr{C}$ if $\mathscr{C}$ is the base of a triangulated monoidal derivator $\mathbb{D}$ (see Example 7.12). It is noteworthy that if we drop the separability assumption on M, we

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can still show that M-$\mathbf{Mod}_{\mathbb{D}}$ is a stable derivator (see Proposition 6.4) but it might not be strong.[1] This allows us to construct examples of derivators that are stable but not strong (see Corollary 7.13). Moreover, it explains why the base of M-$\mathbf{Mod}_{\mathbb{D}}$ might fail to be triangulated: our (potential) inability to turn an incoherent M-linear morphism f to a coherent one is exactly the obstruction to constructing an action of M on the cone of f in a way that is compatible with the triangulation.

1. 2-Monads and their modules

In this section, we introduce some results on the theory of 2-monads and their modules. These will be used later, and in particular in Section 3. We will assume the reader is familiar with some basic 2-category theory; if not, he or she is referred to [18] or [6]. All 2-categories and 2-functors under consideration will be strict. We will denote by **CAT** the 2-category of all categories, functors and natural transformations, and by **2-CAT** the 2-category of 2-categories, 2-functors, and 2-natural transformations. We will ignore the set-theoretic issues with forming **CAT** and **2-CAT**, which can be circumvented by passing to a higher universe.

We begin with the general definition of a monad internal to a 2-category:

Definition 1.1. *Let $\mathcal{K}$ be a 2-category, and let x be an object of $\mathcal{K}$. A **monad** on x consists of a triple $(M, \mu, \mathbb{S})$, where $M : x \to x$ is a 1-cell in $\mathcal{K}$, and $\mu : M^2 \to M$ (the multiplication) and $\mathbb{S} : 1_x \to M$ (the unit) are 2-cells such that the two diagrams below commute:*

$$\begin{array}{ccc} M^3 & \xrightarrow{\mu M} & M^2 \\ {\scriptstyle M\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ M^2 & \xrightarrow[\mu]{} & M \end{array} \qquad \begin{array}{ccccc} M & \xrightarrow{\mathbb{S}M} & M^2 & \xleftarrow{M\mathbb{S}} & M \\ & {\scriptstyle 1_M}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_M} & \\ & & M & & \end{array} \tag{1.2}$$

We will often denote a monad by M, leaving the multiplication and unit out of the notation.

Definition 1.3. *A **2-monad on** a 2-category $\mathcal{K}$ is a monad on $\mathcal{K}$, where $\mathcal{K}$ is viewed as an object of the 2-category **2-CAT**.*

Note that when $\mathcal{K} = \mathbf{CAT}$ Definition 1.1 specializes to the usual definition of a monad in ordinary category theory (see, for instance [19, Chapter VI]). When appropriate we will refer to such a monad as a **classical monad**. In this context, M is an endofunctor of some category, and $\mu, \mathbb{S}$ are natural transformations as in Definition 1.1 making the diagrams (1.2) commute. By contrast, a 2-monad consists of a 2-endofunctor $T : \mathcal{K} \to \mathcal{K}$, with 2-natural transformations for its multiplication and unit. To avoid confusion, we will denote a 2-monad by T and reserve M for a classical monad or (later) for a monad of (pre)derivators.

By forgetting all the 2-structure, it is clear that a 2-monad has an underlying classical monad. We remark that Definition 1.3 can be weakened in various ways. For instance, we could require the multiplication or the unit (or both) to be lax natural transformations instead of 2-natural, even if we still require the 2-functor T to be strict. For the purposes of this paper though, the above definition will be sufficient.

Definition 1.4. *Let T be a 2-monad on a 2-category $\mathcal{K}$. A T-**module** (or T-**algebra**) is a pair (x, λ), where x is an object of $\mathcal{K}$ and $\lambda : Tx \to x$ is a morphism, such that the following diagrams commute:*

[1] In the literature, the axioms for a stable derivator include strongness; we depart from this by calling a derivator stable if it satisfies all axioms in the literature *except* strongness (see Definitions 6.3 and 7.4) and reserving the term "triangulated" for derivators that are both (see Definition 7.5).

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