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AN UNCOUNTABLE FAMILY OF ALMOST NILPOTENT VARIETIES OF POLYNOMIAL GROWTH

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Abstract. A non-nilpotent variety of algebras is almost nilpotent if any proper subvariety is nilpotent. Let the base field be of characteristic zero. It has been shown that for associative or Lie algebras only one such variety exists. Here we present infinite families of such varieties. More precisely we shall prove the existence of

1) a countable family of almost nilpotent varieties of at most linear growth and
2) an uncountable family of almost nilpotent varieties of at most quadratic growth.

1. Introduction

Let F be a field of characteristic zero and $F\{X\}$ the free non associative algebra on a countable set X over F. If $\mathcal{V}$ is a variety of not necessarily associative algebras and $Id(\mathcal{V})$ is the T-ideal of polynomial identities of $\mathcal{V}$, then $F\{X\}/Id(\mathcal{V})$ is the relatively free algebra of countable rank of the variety $\mathcal{V}$. It is well known that in characteristic zero every identity is equivalent to a system of multilinear ones, and an important invariant is provided by the sequence of dimensions $c_n(\mathcal{V})$ of the n-multilinear part of $F\{X\}/Id(\mathcal{V})$, $n = 1, 2, \ldots$. More precisely, for every $n \geq 1$ let P_n be the space of multilinear polynomials in the variables $x_1, \ldots, x_n$. Since char $F = 0$, $F\{X\}/Id(\mathcal{V})$ is determined by the sequence of subspaces $\{P_n/(P_n \cap Id(\mathcal{V}))\}_{n\geq 1}$ and the integer $c_n(\mathcal{V}) = \dim P_n/(P_n \cap Id(\mathcal{V}))$ is called the n-th codimension of $\mathcal{V}$. The growth function determined by the sequence of integers $\{c_n(\mathcal{V})\}_{n\geq 1}$ is the growth of the variety $\mathcal{V}$.

In general a variety $\mathcal{V}$ has overexponential growth, i.e., the sequence of codimensions cannot be bounded by any exponential function. Recall that $\mathcal{V}$ has exponential growth if $c_n(\mathcal{V}) \leq a^n$, for all $n \geq 1$, for some constant a. For instance any variety generated by a finite dimensional algebra has exponential growth. For such varieties the limit $\lim_{n\to\infty} \sqrt[n]{c_n(\mathcal{V})} = exp(\mathcal{V})$, is called the PI-exponent of the variety $\mathcal{V}$, provided it exists.

We say that a variety $\mathcal{V}$ has polynomial growth if there exist constants $\alpha, t \geq 0$ such that asymptotically $c_n(\mathcal{V}) \simeq \alpha n^t$. When $t = 1$ we speak of linear growth and when $t = 2$, of quadratic growth.

Moreover $\mathcal{V}$ has intermediate growth if for any $k > 0$, $a > 1$ there exist constants C_1, C_2, such that for any n the inequalities

$$C_1 n^k < c_n(\mathcal{V}) < C_2 a^n$$

hold. Finally we say that a variety $\mathcal{V}$ has subexponential growth if for any constant B there exists n_0 such that for all $n > n_0$, $c_n(\mathcal{V}) < B^n$. Clearly varieties with polynomial growth or intermediate growth have subexponential growth and it can be shown that varieties realizing each growth can be constructed. For instance a class of varieties of intermediate growth was constructed in [5].

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