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Local uniformization and arc spaces

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ABSTRACT

Let X be a variety over a perfect field k and let X_∞ be its space of arcs. Given a closed subset Z of X, let X_∞^Z denote the subscheme of X_∞ consisting of all arcs centered at some point of Z. We prove that Local Uniformization implies that X_∞^Z has a finite number of irreducible components for each closed subset Z of X. In particular, Local Uniformization implies that $X_\infty^{Sing\,X}$ has a finite number of irreducible components.

1. Introduction

In 1968, J. Nash initiated the study of the space of arcs X_∞ of a (singular) algebraic variety X with the purpose of understanding the resolutions of singularities of X. His work [9] was done shortly after the proof of resolution of singularities in characteristic zero by H. Hironaka [4].

Nash's starting point is the following: let X be a variety over a perfect field k and suppose that there exists a resolution of singularities $\pi : Y \to X$. For every irreducible component E of the exceptional locus of π, let us consider the *Nash family of arcs* N_E consisting of the Zariski closure of the image of the set of arcs on Y which are centered in some point of E. He observes that each N_E is irreducible and, moreover, N_E only depends on the divisorial valuation defined by E. Besides, due to the properness of π, every arc in $X_\infty \setminus (\operatorname{Sing} X)_\infty$ which is centered in some point of the singular locus of X belongs to some of the N_E's. That is, the space of arcs X_∞^{Sing} centered in $\operatorname{Sing} X$ decomposes as

$$X_\infty^{Sing} = \cup_E N_E \cup (\operatorname{Sing} X)_\infty.$$

From this, and arguing by induction on $\dim X$, one deduces that the number of irreducible components of X_∞^{Sing} is finite (see [9] or [5,10]).

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No other proof of the previous result is known until now. In particular, it is unknown whether the number of irreducible components of X_∞^{Sing} is finite or not if char $k = p > 0$ and $\dim X \geq 4$, since resolution of singularities is still an open problem in this situation. The purpose of this note is to prove that Hironaka's resolution of singularities can be substituted by the weaker notion of Zariski's generalized Local Uniformization along valuations in Nash's argument.

More precisely, let X be an irreducible variety over a perfect field k, and $Z \subseteq X$ be a Zariski closed subset, let X_∞^Z be the space of arcs on X centered at some point of Z. We consider valuation rings $(\mathcal{O}_v, M_v, k_v)$ of the field of fractions $k(X)$ of X. Say that $LU(X, \zeta)$ holds for $\zeta \in X$ an arbitrary point if for any valuation ring $\mathcal{O}_v$ dominating $\mathcal{O}_{X,\zeta}$, there exists a finitely generated algebra

$$R = \mathcal{O}_{X,\zeta}[f_1, \ldots, f_r] \subseteq \mathcal{O}_v$$

such that $R_{M_v \cap R}$ is a regular local ring. We prove the following result:

Main Result. (Corollary 3.7) *Let $X|k$ be a k-variety (not necessarily irreducible) and $Z \subseteq X$. Assume that, for every $z \in Z$ and for every irreducible subvariety $V \subseteq X$ with $z \in V$, Local Uniformization holds on V at z, i.e. $LU(V, z)$ holds. Then X_∞^Z has a finite number of irreducible components.*

The proof of this result relies heavily on concepts developed by Zariski (chapter VI of [12]) in his study of the space of valuation rings of a function field: composition of valuations, quasi-compactness, description of birational correspondence in terms of valuations.

2. Preliminaries

In this section we set the notation and recall some known properties of the space of arcs that will be used in this work. We then recall some basic facts from valuation theory: centers, domination and composition of valuations.

2.1. Arc spaces. Let k be a perfect field and let X be a *variety over* k, i.e. X is a reduced separated k-scheme of finite type. Let X_∞ denote the *space of arcs of* X: It is a k-scheme whose K-rational points are the K-arcs on X, i.e. the k-morphisms $\operatorname{Spec} K[[t]] \to X$, for any field extension $k \subseteq K$. More precisely, $X_\infty := \lim_{\leftarrow} X_n$ where, for $n \in \mathbb{N}$, X_n is the k-scheme of finite type whose K-rational points are the K-arcs of order n on X, i.e. the k-morphisms $\operatorname{Spec} K[[t]]/(t)^{n+1} \to X$.

For instance, the space of arcs of the affine space $\mathbb{A}_k^N = \operatorname{Spec} k[x_1, \ldots, x_N]$ is

$$(\mathbb{A}_k^N)_\infty = \operatorname{Spec} k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots]$$

where for $n \geq 0$, $\underline{X}_n = (X_{1;n}, \ldots, X_{N;n})$ is an N-uple of variables. For any $f \in k[x_1, \ldots, x_N]$, let $\sum_{n=0}^\infty F_n \ t^n$ be the formal series expansion of $f(\sum_n \underline{X}_n \ t^n)$, hence $F_n \in k[\underline{X}_0, \ldots, \underline{X}_n]$. If $X \subseteq \mathbb{A}_k^N$ is affine, and $I_X \subset k[x_1, \ldots, x_N]$ is the ideal defining X in $\mathbb{A}_k^N$, then we have $X_\infty = \operatorname{Spec} k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots] \ / \ (\{F_n\}_{n \geq 0, f \in I_X})$.

We denote by $j_n : X_\infty \to X_n$, $n \geq 0$, the natural projections. We also denote by $j : X_\infty \to X$ the projection j_0. Given a closed subset Z of X, we denote by X_∞^Z the subscheme $j^{-1}(Z)$ of X_∞.

Given $P \in X_\infty$, with residue field $\kappa(P)$, we denote by $h_P : \operatorname{Spec} \kappa(P)[[t]] \to X$ the induced $\kappa(P)$-arc on X. The image x_0 in X of the closed point of $\operatorname{Spec} \kappa(P)[[t]]$, i.e. $x_0 = j(P)$, is called the *center* of P. The image x_η of the generic point of $\operatorname{Spec} \kappa(P)[[t]]$ is the generic point of $\operatorname{Im} h_P$. Then, h_P induces a morphism of k-algebras $h_P^\sharp : \mathcal{O}_{X,x_0} \to \kappa(P)[[t]]$, or an injective morphism $h_P^\sharp : \mathcal{O}_{\overline{\{x_\eta\}},x_0} \to \kappa(P)[[t]]$. We denote by v_P the function $\operatorname{ord}_t h_P^\sharp : \mathcal{O}_{X,x_0} \to \mathbb{N} \cup \{\infty\}$, which defines a valuation of $k(x_\eta)$.

The scheme X_∞ is not of finite type over k if $\dim X > 0$. However it satisfies several finiteness properties. The following result refers to one of them. It is proved in [7] chap. IV, prop. 10 if char $k = 0$ and in [3] cor. 1.28, [10] th. 2.9 for any perfect field k.

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