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Maps of Mori dream spaces

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A B S T R A C T

Let $\phi\colon X \to Y$ be a map of $\mathbb{Q}$-factorial Mori dream spaces. We prove that there is a unique *Cox lift* $\Phi\colon \mathcal{X} \to \mathcal{Y}$ of Mori dream stacks coming from a homogeneous homomorphism $\mathcal{R}(Y) = \mathcal{R}(\mathcal{Y}) \to \mathcal{R}(\mathcal{X})$, where $\mathcal{Y}$ is a canonical stack of Y and $\mathcal{X}$ is obtained from X by root constructions, and ϕ is induced from Φ by passing to coarse moduli spaces. We also apply this techniques to show that a Mori dream quotient stack is obtained by roots from its canonical stack.

1. Introduction

Given a map $\phi\colon X \to Y$ of $\mathbb{Q}$-factorial Mori dream spaces, one can ask whether this map lifts to a homogeneous homomorphism $\mathcal{R}(Y) \to \mathcal{R}(X)$ of Cox rings. As soon as Y is singular, such a homomorphism needs not to exist, as pulling back Weil divisors is not well-defined.

In [9], Gavin Brown and Jarosław Buczyński show how to lift rational maps between toric varieties to (multi-valued) maps between their respective Cox rings. In this article, we show that this construction has a geometrical interpretation involving quotient stacks and works well for Mori dream spaces. This leads to our main result; see Theorem 3.2 for a more precise and more general version.

Main Theorem. *Let $\phi\colon X \to Y$ be a map between $\mathbb{Q}$-factorial Mori dream spaces where X is complete with factorial Cox ring $\mathcal{R}(X)$. There is a Mori dream quotient stack $\mathcal{X}$ with coarse moduli space X with the following properties:*

1. *The stack $\mathcal{X}$ is built from the canonical Mori dream stack $\mathcal{X}^{can}$ of X by root constructions with prime divisors and line bundles.*

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2. *There is a homogeneous morphism* $\Phi^* \colon \mathcal{R}(Y) \to \mathcal{R}(\mathcal{X})$ *such that the following diagrams commute:*

$$\begin{tikzcd} \mathcal{R}(Y) \arrow[r, "\Phi^*"] & \mathcal{R}(\mathcal{X}) \\ \mathcal{R}(Y;\mathrm{Pic}(Y)) \arrow[u] \arrow[r, "\phi^*"'] & \mathcal{R}(X) \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X} \arrow[r, "\Phi"] \arrow[d] & \mathcal{Y}^{can} \arrow[d] \\ X \arrow[r, "\phi"'] & Y \end{tikzcd}$$

where $\mathcal{Y}^{can}$ *is the canonical Mori dream stack of* Y*.*

3. *The stack* $\mathcal{X}$ *is minimal: given any Mori dream quotient stack* $\mathcal{X}'$ *that satisfies* (2)*, then the map* $\mathcal{X}' \to X$ *to the coarse moduli space factors through* $\mathcal{X}$*.*

We call $\Phi\colon \mathcal{X} \to \mathcal{Y}^{can}$ *the* Cox lift *of the map* $\phi\colon X \to Y$*.*

The following example illustrates this theorem.

Example 1.1 *(cf. [9, Ex. 1.1.1]).* Let ξ be a primitive k-th root of unity acting on $\mathbb{A}^2$ by $\xi \cdot (x,y) = (\xi^a x, \xi^b y)$. The quotient $\mathbb{A}^2/\mu_k$ is then called a $\frac{1}{k}(a,b)$-singularity. Consider the $\frac{1}{2}(1,1)$-singularity $\mathbb{A}^2/\mu_2$. In the coordinates $\mathbb{A}^2 = \operatorname{Spec} \Bbbk[x,y]$, the quotient is given as

$$\mathbb{A}^2/\mu_2 = \operatorname{Spec} \Bbbk[x,y]^{\mathbb{Z}_2} = \operatorname{Spec} \Bbbk[x^2, xy, y^2] = \operatorname{Spec} \Bbbk[u,v,w]/uw - v^2.$$

Geometrically, this is a double cone with an isolated singularity at 0. Let $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[t]$ be a Weil divisor passing through the singular origin. For example, we can choose the embedding $\phi\colon \mathbb{A}^1 \hookrightarrow \mathbb{A}^2/\mu_2$ as in Fig. 1 given in terms of the coordinate rings as

$$\begin{array}{rccc} \phi^*\colon & \Bbbk[x^2, xy, y^2] & \to & \Bbbk[t] \\ & x^2 & \mapsto & t \\ & xy, y^2 & \mapsto & 0 \end{array}$$

The Cox ring of $Y = \mathbb{A}^2/\mu_2$ is $\mathcal{R}(Y) = \Bbbk[x,y]$ graded by $\mathrm{Cl}(Y) = \mathbb{Z}_2$, which is given by $\deg(x) = \deg(y) = 1$. The Cox ring of $X = \mathbb{A}^1$ is again the ring $\mathcal{R}(X) = \Bbbk[t]$ which is graded trivially since $\mathrm{Cl}(X) = 0$. As the authors of [9] point out, the best way to lift the map ϕ^* to the Cox rings is

$$\begin{array}{rccc} \Phi^*\colon & \Bbbk[x,y] & \to & \Bbbk[\sqrt{t}] \\ & x & \mapsto & \sqrt{t} \\ & y & \mapsto & 0 \end{array}$$

Here $\sqrt{t}$ should mean that we consider an extension of $\Bbbk[t]$, namely, the ring $\Bbbk[t][z]/(z^2 - t) = \Bbbk[\sqrt{t}]$, where we write suggestively $\sqrt{t} := z$. As shown in [9], this map in homogeneous coordinates has several nice properties, like the images of points or the pullback of divisors can be easily computed.

Our main observation is that the map of Cox rings above is inherently a map between toric stacks. The singular space $\mathbb{A}^2/\mu_2$ is the coarse moduli space of the smooth quotient stack

$$\mathcal{Y} = \left[\operatorname{Spec} \Bbbk[x,y] \middle/ \mu_2\right].$$

Although the origin is a smooth point of $\mathcal{Y}$, it is still different from all other points, since it has the non-trivial stabiliser μ_2.

So if we consider again the divisor as above through the origin, the corresponding point on the embedded divisor should also have the μ_2-stabiliser. We can obtain such a stack by performing a root construction at the origin on $X = \mathbb{A}^1$, to get $\mathcal{X} = \sqrt[2]{0/\mathbb{A}^1}$. This stack has the following description as a quotient

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