ARTICLE IN PRESS
JPAA:5704

Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

The connection between $R^*(\mathcal{C}_g^n)$ and $R^*(\mathcal{M}_{g,n}^{rt})$

Mehdi Tavakol

IBS Center for Geometry and Physics, Republic of Korea

A R T I C L E   I N F O

Article history:
Received 21 January 2015
Received in revised form 11 March 2017
Available online xxxx
Communicated by R. Vakil

A B S T R A C T

We study the connection between tautological classes on the moduli spaces $\mathcal{C}_g^n$ and $\mathcal{M}_{g,n}^{rt}$. A conjectural connection between tautological relations on $\mathcal{C}_g^n$ and $\mathcal{M}_{g,n}^{rt}$ is described. The analogue of this conjecture for the Gorenstein quotients of tautological rings is proved.

0. Introduction

Let $\mathcal{M}_{g,n}^{rt}$ be the moduli space of stable n-pointed curves of genus $g > 1$ with rational tails. This space is a partial compactification of the space $\mathcal{M}_{g,n}$, classifying smooth curves of genus g together with n ordered distinct points. It sits inside the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$, which classifies all stable n-pointed curves of genus g. We also consider the space $\mathcal{C}_g^n$ classifying smooth curves of genus g with not necessarily distinct n ordered points. There is a natural proper map from $\mathcal{M}_{g,n}^{rt}$ to $\mathcal{C}_g^n$ which contracts all rational components. Tautological classes on these spaces are natural algebraic cycles reflecting the geometry of curves. Their definition and analysis on $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ was started by Mumford in the seminal paper [5]. These were later generalized to pointed spaces by Faber and Pandharipande [2]. In this short note we study the connection between tautological classes on $\mathcal{C}_g^n$ and $\mathcal{M}_{g,n}^{rt}$. We show that there is a natural filtration on the tautological ring of $\mathcal{M}_{g,n}^{rt}$ consisting of $g-2+n$ steps. A conjectural dictionary between tautological relations on $\mathcal{C}_g^n$ and $\mathcal{M}_{g,n}^{rt}$ is presented. We prove the analogue version of our conjecture for the Gorenstein quotients of tautological rings. It is well-known that the tautological group of $\mathcal{M}_{g,n}^{rt}$ is one dimensional in degree $g-2+n$ and it vanishes in higher degrees. In [3] the authors predicted that these should be deductible from the main theorem of Looijenga [4]. We show that both statements follow from the result of Looijenga on $\mathcal{C}_g^n$ and a result of Faber.

Conventions 0.1. We consider algebraic cycles modulo rational equivalence. All Chow groups are taken with $\mathbb{Q}$-coefficients.

E-mail address: mehdi@ibs.re.kr.

http://dx.doi.org/10.1016/j.jpaa.2017.06.019
0022-4049/

Please cite this article in press as: M. Tavakol, The connection between $R^*(\mathcal{C}_g^n)$ and $R^*(\mathcal{M}_{g,n}^{rt})$, J. Pure Appl. Algebra (2017), http://dx.doi.org/10.1016/j.jpaa.2017.06.019

1. Tautological algebras

1.1. The moduli spaces

Let $\pi : \mathcal{C}_g \to \mathcal{M}_g$ be the universal smooth curve of genus $g > 1$. For an integer $n > 0$ the n-fold fiber product of $\mathcal{C}_g$ over $\mathcal{M}_g$ is denoted by $\mathcal{C}_g^n$. It parameterizes smooth curves of genus g together with n ordered points. We also consider the space $\mathcal{M}_{g,n}^{rt}$ which classifies stable n-pointed curves of genus g with rational tails. This moduli space classifies stable curves of arithmetic genus g consisting of a unique component of genus g. These conditions imply that all other components are rational. The marked points belong to the smooth locus of the curve. Marked points and nodal points are called special. By the stability condition we require that every rational component has at least 3 special points. As a result the corresponding moduli point has finitely many automorphisms and the associated stack is of Deligne–Mumford type. Recall that the boundary $\mathcal{M}_{g,n}^{rt} \setminus \mathcal{M}_{g,n}$ is a divisor with normal crossings. It is a union of irreducible divisor classes D_I parameterized by subsets I of the set $\{1, \ldots, n\}$ having at least 2 elements. The generic point of the divisor D_I corresponds to a nodal curve with 2 components. One of its components is of genus g and the other component is rational. The set I refers to the markings on the rational component.

1.2. Tautological classes

Tautological classes are natural algebraic cycles on the moduli space. Their original definition on the space $\mathcal{M}_g$ and its compactification $\overline{\mathcal{M}}_g$ is given by Mumford in [5]. The definition of tautological algebras of moduli spaces of pointed curves is due to Faber and Pandharipande [2]. We will recall the definition below.

Definition 1.1. Let g, n be integers so that $2g - 2 + n > 0$. The system of tautological rings is defined to be the set of smallest $\mathbb{Q}$-subalgebras $R^*(\overline{\mathcal{M}}_{g,n})$ of the Chow rings $A^*(\overline{\mathcal{M}}_{g,n})$ which is closed under push-forward via all maps forgetting points and all gluing maps.

Definition 1.2. The tautological ring $R^*(\mathcal{M}_{g,n}^{rt})$ of $\mathcal{M}_{g,n}^{rt}$ is the image of $R^*(\overline{\mathcal{M}}_{g,n})$ via the restriction map.

Recall that there is a proper map $p : \mathcal{M}_{g,n}^{rt} \to \mathcal{C}_g^n$ which contracts all rational components of the curve.

Definition 1.3. The tautological ring $R^*(\mathcal{C}_g^n)$ of $\mathcal{C}_g^n$ is defined as the image of $R^*(\mathcal{M}_{g,n}^{rt})$ via the push-forward map $p_* : A^*(\mathcal{M}_{g,n}^{rt}) \to A^*(\mathcal{C}_g^n)$.

There is a more explicit presentation of the tautological ring of $\mathcal{C}_g^n$ and $\mathcal{M}_{g,n}^{rt}$. Let $\pi : \mathcal{C}_g \to \mathcal{M}_g$ be the universal curve of genus g as before. Denote by ω its relative dualizing sheaf. Its class in the Chow group $A^1(\mathcal{C}_g)$ of $\mathcal{C}_g$ is denoted by K. This defines the class K_i for every $1 \leq i \leq n$ on $\mathcal{C}_g^n$. We also denote by $d_{i,j}$ the class of the diagonal for $1 \leq i < j \leq n$. The kappa class κ_i is the push-forward class $\pi_*(K^{i+1})$. According to the vanishing results proven by Looijenga [4] the class κ_i is zero when $i > g - 2$. We denote the pull-back of these classes to $\mathcal{M}_{g,n}^{rt}$ along the contraction map p by the same letters. The proof of the following fact is left to the reader:

Lemma 1.4. *(1) The tautological ring of $\mathcal{C}_g^n$ is the $\mathbb{Q}$-subalgebra of its Chow ring $A^*(\mathcal{C}_g^n)$ generated by kappa classes κ_i for $1 \leq i \leq g - 2$ together with the classes K_i and $d_{i,j}$ for $1 \leq i < j \leq n$.*

(2) The tautological ring of $\mathcal{M}_{g,n}^{rt}$ is the $\mathbb{Q}$-subalgebra of $A^*(\mathcal{M}_{g,n}^{rt})$ generated by κ_i for $1 \leq i \leq g - 2$, the divisor classes $K_i, d_{i,j}$ and D_I for $1 \leq i < j \leq n$ and subsets I of the set $\{1, \ldots, n\}$ having at least 3 elements.