Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Milnor K-theory of complete discrete valuation rings with finite residue fields

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A R T I C L E   I N F O

Article history:
Received 15 June 2017
Available online xxxx
Communicated by C.A. Weibel

A B S T R A C T

Consider a complete discrete valuation ring $\mathcal{O}$ with quotient field F and finite residue field. Then the inclusion map $\mathcal{O} \hookrightarrow F$ induces a map $\hat{\mathrm{K}}^{\mathrm{M}}_{*}\mathcal{O} \to \hat{\mathrm{K}}^{\mathrm{M}}_{*}F$ on improved Milnor K-theory. We show that this map is an isomorphism in degrees bigger or equal to 3. This implies the Gersten conjecture for improved Milnor K-theory for $\mathcal{O}$. This result is new in the p-adic case.

1. Introduction

Let $\hat{\mathrm{K}}^{\mathrm{M}}_{*}$ denote the improved Milnor K-theory as introduced by Gabber [2] and mainly developed by Kerz [8]. For fields, it coincides with the usual Milnor K-theory. For $n \geq 1$ and a discrete valuation ring $\mathcal{O}$ with quotient field F and residue field κ, there is the so-called Gersten complex

$$0 \longrightarrow \hat{\mathrm{K}}^{\mathrm{M}}_{n}\mathcal{O} \longrightarrow \hat{\mathrm{K}}^{\mathrm{M}}_{n}F \longrightarrow \hat{\mathrm{K}}^{\mathrm{M}}_{n-1}\kappa \longrightarrow 0.$$

This complex is known to be right-exact. Is it also exact on the left side? In the equicharacteristic case this was shown by Kerz [8, Prop. 10 (8)]. As a slight extension to a special case of the mixed characteristic case, we show the exactness on the left side by proving the following result (Theorem 4.2).

Theorem. *Let $\mathcal{O}$ be a complete discrete valuation ring with quotient field F and finite residue field. Then for $n \geq 3$ the inclusion map $\iota\colon \mathcal{O} \hookrightarrow F$ induces an isomorphism*

$$\iota_{*}\colon \hat{\mathrm{K}}^{\mathrm{M}}_{n}\mathcal{O} \xrightarrow{\cong} \hat{\mathrm{K}}^{\mathrm{M}}_{n}F$$

on improved Milnor K-theory.

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[1] The author is supported by the DFG GRK 1692 “Curvature, Cycles and Cohomology”.

http://dx.doi.org/10.1016/j.jpaa.2017.07.002
0022-4049/

In the p-adic case this is new. We prove the theorem as follows: We show that $\hat{\mathrm{K}}_n^{\mathrm{M}} \mathcal{O}$ is a divisible abelian group for $n \geq 3$ (Theorem 3.7) using results from the appendix of Milnor's book [11]. Combining this with the unique divisibility of $\hat{\mathrm{K}}_n^{\mathrm{M}} F$ (proved by Sivitskii [14]) and a comparison between algebraic and Milnor K-theory (proved by Nesterenko and Suslin [12]) yields the theorem.

Acknowledgements. I am grateful to my advisor Moritz Kerz for proposing me this interesting topic. Also, I thank Morten Lüders for useful comments and fruitful discussions and Thomas Fiore for having a look at the language. I want to thank my unknown referee from AKT (though my submission was finally declined). Their reports helped a lot in clarifying the presentation of this paper's content. Last, but not least, I thank Chuck Weibel for helpful comments on the presentation of the paper's content.

2. Milnor K-theory

Definition 2.1. Let A be a commutative ring with unit, $\mathrm{T}_* A^\times$ be the (non-commutative) tensor algebra of $A^\times$ over $\mathbf{Z}$, and $\mathrm{StR}_* A^\times$ the homogeneous ideal of $\mathrm{T}_* A^\times$ which is generated by the set $\{x \otimes y \in \mathrm{T}_2 A^\times \mid x + y = 1\}$ (the so-called Steinberg relations). Define the Milnor K-theory of A to be the graded ring

$$\mathrm{K}_*^{\mathrm{M}} A := \mathrm{T}_* A^\times / \mathrm{StR}_* A^\times.$$

For $x_1, \ldots, x_n \in A^\times$ let $\{x_1, \ldots, x_n\}$ denote the image of $x_1 \otimes \ldots \otimes x_n$ under the natural homomorphism $\mathrm{T}_n A^\times \to \mathrm{K}_n^{\mathrm{M}} A$. Evidently, this yields a functor from commutative rings to abelian groups.

This notion behaves well if A is a field or a local ring with infinite residue field [7]. But some nice properties do not hold for arbitrary commutative rings (e.g. that the natural map to algebraic K-theory is an isomorphism in degree 2). For local rings, this lack is repaired by a generalisation due to Gabber [2], the improved Milnor K-theory, which was mainly developed by Kerz [8].

Definition 2.2. Let A be a local ring and $n \in \mathbf{N} := \mathbf{Z}_{\geq 0}$. The subset

$$S := \Big\{ \sum_{\underline{i} \in \mathbf{N}^n} a_{\underline{i}} \cdot \underline{t}^{\underline{i}} \in A[t_1, \ldots, t_n] \;\Big|\; \langle a_{\underline{i}} \mid \underline{i} \in \mathbf{N}^n \rangle = A \Big\}$$

of $A[t_1, \ldots, t_n]$ is multiplicatively closed, where $\underline{t}^{\underline{i}} = t_1^{i_1} \cdot \ldots \cdot t_n^{i_n}$. Define the ring of rational functions (in n variables) to be $A(t_1, \ldots, t_n) := S^{-1} A[t_1, \ldots, t_n]$. We obtain maps $\iota \colon A \to A(t)$ and $\iota_1, \iota_2 \colon A(t) \to A(t_1, t_2)$ by mapping t respectively to t_1 or t_2.

For $n \geq 0$ we define the n-th improved Milnor K-theory of A to be

$$\hat{\mathrm{K}}_n^{\mathrm{M}} A := \ker\big[\mathrm{K}_n^{\mathrm{M}} A(t) \xrightarrow{\delta_n^{\mathrm{M}}} \mathrm{K}_n^{\mathrm{M}} A(t_1, t_2)\big],$$

where $\delta_n^{\mathrm{M}} := \mathrm{K}_n^{\mathrm{M}}(\iota_1) - \mathrm{K}_n^{\mathrm{M}}(\iota_2)$. By definition, we have an exact sequence

$$0 \longrightarrow \hat{\mathrm{K}}_n^{\mathrm{M}} A \xrightarrow{\iota_*} \mathrm{K}_n^{\mathrm{M}} A(t) \xrightarrow{\delta_n^{\mathrm{M}}} \mathrm{K}_n^{\mathrm{M}} A(t_1, t_2).$$

In particular, for $n = 0$ we have $\hat{\mathrm{K}}_0^{\mathrm{M}} = \ker\big(\mathbf{Z} \xrightarrow{0} \mathbf{Z}\big) = \mathbf{Z}$. As a direct consequence of the construction we obtain a natural homomorphism

$$\mathrm{K}_*^{\mathrm{M}} A \longrightarrow \hat{\mathrm{K}}_*^{\mathrm{M}} A.$$

We state some facts about Milnor K-theory of local rings.

Please cite this article in press as: C. Dahlhausen, Milnor K-theory of complete discrete valuation rings with finite residue fields, J. Pure Appl. Algebra (2017), http://dx.doi.org/10.1016/j.jpaa.2017.07.002