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AN ENSEMBLE OF IDEMPOTENT LIFTING HYPOTHESES

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Abstract. Lifting idempotents modulo ideals is an important tool in studying the structure of rings. This paper lays out the consequences of lifting other properties modulo ideals, including lifting of von Neumann regular elements, lifting isomorphic idempotents, and lifting conjugate idempotents. Applications are given for IC rings, perspective rings, and Dedekind-finite rings, which improve multiple results in the literature. We give a new characterization of the class of exchange rings; they are rings where regular elements lift modulo all left ideals.

We also uncover some hidden connections between these lifting properties. For instance, if regular elements lift modulo an ideal, then so do isomorphic idempotents. The converse is true when units lift. The logical relationships between these and several other important lifting properties are completely characterized. Along the way, multiple examples are developed that illustrate limitations to the theory.

1. Introduction

Given a ring R, an element $a \in R$ is said to be *von Neumann regular* if $a = aba$ for some $b \in R$. In such a case we call b an *inner inverse* of a. It is common to refer to von Neumann regular elements simply as regular elements, and we will follow that convention in this paper. Similarly, a *regular ring* is one where each of its elements is regular. These rings were first introduced in the work of von Neumann on continuous geometries. It is well known that a ring R is regular if and only if every finitely generated left (or right) ideal is generated by an idempotent, if and only if all R-modules are flat (i.e. R has weak dimension 0).

Even when working on the level of elements rather than rings, regularity captures important ring-theoretic and module-theoretic information. To give one example, if R is the endomorphism ring of some module (for instance, by identifying R in the natural way with $\mathrm{End}(R_R)$), then regular elements correspond to those endomorphisms whose kernels and images are direct summands [18, Exercise 4.14A_1]. As each direct sum decomposition of a module is determined by an idempotent in the endomorphism ring, we see that regular elements are intricately connected to idempotents. This relationship will be made more precise after we introduce some further terminology in the next section.

Besides the reasons listed above, there is one further justification for interest in regular rings; idempotents lift modulo all left and right (and two-sided) ideals in such rings [23, Proposition 1.6], and consequently we can convert information about the projective modules over a factor ring into information on projective modules over the original ring. The purpose of this paper is to study such lifting more generally, and so we introduce some of the basic definitions here. Letting I be a one-sided ideal of a ring R, we say that $x \in R$ is

$$\left.\begin{array}{l} \textit{regular} \\ \textit{idempotent} \\ \textit{a unit} \end{array}\right\} \text{ modulo } I \text{ if } \left\{\begin{array}{l} \exists\, y \in R:\ x - xyx \in I \\ x - x^2 \in I \\ \exists\, y \in R:\ 1 - xy, 1 - yx \in I. \end{array}\right.$$

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