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Definable categories

Amit Kuber [1], Jiří Rosický *

Department of Mathematics and Statistics, Masaryk University, Faculty of Sciences, Kotlářská 2, 611 37 Brno, Czech Republic

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ABSTRACT

We introduce the notion of a definable category–a category equivalent to a full subcategory of a locally finitely presentable category that is closed under products, directed colimits and pure subobjects. Definable subcategories are precisely the finite-injectivity classes. We prove a 2-duality between the 2-category of small exact categories and the 2-category of definable categories, and provide a new proof of its additive version. We further introduce a third vertex of the 2-category of regular toposes and show that the diagram of 2-(anti-)equivalences between three 2-categories commutes; the corresponding additive triangle is well-known.

1. Introduction

This paper belongs to the realm of categorical logic where one studies the interplay between syntax and semantics using the language of category theory, that is usually presented in the form of a dual adjunction as below:

$$(-)\text{-Mod}(\mathrm{Set}) : \mathrm{Theories}^{op} \leftrightarrows \text{Exactness properties} : \mathrm{Exact}(-, \mathrm{Set}) \tag{1}$$

In Makkai's terminology, for a given fragment of first-order logic, such an adjunction is a consequence of a 'logical doctrine', namely the fact that certain limits, colimits or combinations thereof commute with/distribute over others in the base category Set of sets. The above notion of 'exactness property' on the semantics side precisely captures these commutativity/distributivity conditions, and the exact functors preserve all combinations of limits and colimits present in the exact categories.

For lex (i.e., finitely complete) categories, the Gabriel–Ulmer duality [11] is a full 2-duality between the 2-category LEX of small lex categories, lex functors and natural transformations on the one hand, and

* Corresponding author.
E-mail addresses: askuber@iitk.ac.in (A. Kuber), rosicky@math.muni.cz (J. Rosický).
[1] Current address: Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, Kalyanpur, Kanpur-208016, India.

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the 2-category LFP of locally finitely presentable categories, finitary right adjoint functors and natural transformations on the other:

$$\mathrm{Lex}(-, \mathrm{Set}) : \mathrm{LEX}^{op} \leftrightarrows \mathrm{LFP} : (-)_{\mathrm{fp}}$$

Using [1, Corollary 4.7] it can be shown that, for a locally finitely presentable category $\mathcal{K}$, a functor $F : \mathcal{K} \to \mathrm{Set}$ preserves all limits and directed colimits (in notation, $F \in (\mathcal{K}, \mathrm{Set})^{\lim\to}$) iff there is $K \in \mathcal{K}_{\mathrm{fp}}$ such that $F \simeq \mathcal{K}(K, -)$ (also see [17, p.101]). Thus the above duality can be rewritten in the following form:

$$\mathrm{Lex}(-, \mathrm{Set}) : \mathrm{LEX}^{op} \leftrightarrows \mathrm{LFP} : (-, \mathrm{Set})^{\lim\to} \tag{2}$$

Cartesian logic is the internal logic of lex categories (see [12, Definition D1.3.4] for the full syntactic definition). For this fragment of first-order logic the 2-adjunction of (1) restricts to the full 2-duality of (2). In order to achieve this, one needs to characterize the categories in the image of the 2-functors in the adjunction given by (1). The 'logical doctrine' for cartesian logic states that finite limits commute with all limits as well as with directed colimits in Set. The 'exactness property' for this logic is captured by precontinuous categories of [1], namely the categories having all limits and directed colimits in which directed colimits commute with finite limits and products distribute over directed colimits. The locally finitely presentable categories which are models of cartesian theories are precisely the precontinuous categories satisfying a smallness condition (see [1, Theorem 5.8]). On the syntactic side, a cartesian theory can be recovered from the category of its models up to Morita equivalence; such Morita equivalence classes are in one-to-one correspondence with small lex categories [12, §D1.4.9].

1.1. The regular case: syntax–semantics duality

One of the main contributions of this paper is Theorem 3.2.5, the following analogue of (2) for regular logic–a fragment of first-order logic where formulas are constructed using truth, finitary conjunctions and existential quantifiers only (Definition 5.3.3).

$$\mathrm{Reg}(-, \mathrm{Set}) : \mathbb{EX}^{op} \leftrightarrows \mathbb{DEF} : (-, \mathrm{Set})^{\Pi\to} \tag{3}$$

The most important new concept in this paper is that of a *definable category* which we borrowed from the additive world. In that context the term 'definable category' first appeared independently in [9] and [15] at around the same time. This notion is 'relative' in nature; such categories are (equivalent to) subcategories of locally finitely presentable categories closed under products, directed colimits and pure subobjects. It is an open question to find an 'absolute' characterization of definable categories, where one provides a complete list of verifiable category-theoretic properties. The 2-category $\mathbb{DEF}$ has definable categories as objects, interpretation functors (i.e., functors preserving products and directed colimits) as 1-morphisms and natural transformations as 2-morphisms. The doctrine for regular logic is the statement that finite limits and coequalizers commute with products and directed colimits in Set. The exactness properties for regular logic are captured by the concept of predefinable categories introduced in Section 2.2; they are categories with products and directed colimits, and where products distribute over directed colimits.

On the left side of the duality is the 2-category $\mathbb{EX}$ of small (Barr-)exact categories, regular functors and natural transformations. Morita equivalence classes of regular theories are in one-to-one correspondence with exact categories. The concept of an exact category was introduced by Barr in [5] with the intention to codify what is common in Set and the category Ab of abelian groups. The existence of the exact completion of a lex category was shown by Carboni and Celia Magno in [8]. One half of the duality (3) was proved by Makkai as (the finitary version of) [18, Theorem 5.1] under the name of 'strong conceptual completeness theorem' for regular logic, where one recovers (the Morita equivalence class of) a theory from its category