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Six-dimensional product Lie algebras admitting integrable complex structures

Andrzej Czarnecki *, Marcin Sroka

Jagiellonian University, Lojasiewicza 6, 30-348 Krakow, Poland

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ABSTRACT

We classify the 6-dimensional Lie algebras of the form $\mathfrak{g} \times \mathfrak{g}$ that admit an integrable complex structure. We also endow a Lie algebra of the kind $\mathfrak{o}(n) \times \mathfrak{o}(n)$ $(n \geq 2)$ with such a complex structure. The motivation comes from geometric structures à la Sasaki on $\mathfrak{g}$-manifolds.

1. Introduction

A complex structure on a Lie algebra $\mathfrak{h}$ is an endomorphism $\mathcal{J} : \mathfrak{h} \to \mathfrak{h}$ such that $\mathcal{J}^2 = -id$. It corresponds to a left invariant almost complex structure on any Lie group H with $T_eH = \mathfrak{h}$. We say that a complex structure $\mathcal{J}$ on $\mathfrak{h}$ is integrable if

$$N(v, w) := [v, w] + \mathcal{J}[\mathcal{J}v, w] + \mathcal{J}[v, \mathcal{J}w] - [\mathcal{J}v, \mathcal{J}w] = 0$$

for any $v, w \in \mathfrak{h}$. By the Newlander–Nirenberg theorem, via a left invariant trivialisation of the tangent bundle of H, this condition is likewise equivalent to the integrability of the corresponding left invariant almost complex structure on H.

Classification of integrable complex structures on real Lie algebras is a well established problem, cf. a summary of results on their existence in [9]. In dimension 6, the question is settled only for special – abelian – complex structures, cf. [1], and for nilpotent algebras, cf. [2,7]. The present paper focuses on a different class of 6-dimensional Lie algebras that split as a product $\mathfrak{g} \times \mathfrak{g}$ for a 3-dimensional Lie algebra $\mathfrak{g}$. We identify all such algebras admitting integrable complex structures. This problem was studied in the special cases of $\mathfrak{o}(3) \times \mathfrak{o}(3)$ and $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$ by Magnin in [4,5], where he also classified all possible

* Corresponding author.
E-mail addresses: andrzejczarnecki01@gmail.com (A. Czarnecki), marcin.sroka@student.uj.edu.pl (M. Sroka).

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integrable complex structures. Such a complete description will not be our concern here, but we note that the classification given in [2] covers types (1), (2), and (3) of our Proposition 1.

Our main motivation comes from differential geometry. As explained in detail in [3], the existence of complex structures on 6-dimensional product algebras has immediate applications to the recently developed theory of non-abelian, higher-dimensional structures à la Sasaki. To get an idea of the problem, consider a 3-dimensional Lie group G acting freely on an odd-dimensional manifold M. Suppose that this action preserves some transverse complex structure – a complex structure on the sub-bundle ν transverse to the orbits. We cannot hope to extend this complex structure to the whole tangent bundle – since the dimension is odd – but an interesting problem is to extend it to the $T(M \times G)$. Since the tangent space at a point x in that product splits (as a vector space) $T_x(M \times G) = \nu_x \times \mathfrak{g} \times \mathfrak{g}$, this rises – and reduces to – the question of finding an integrable complex structure on $\mathfrak{g} \times \mathfrak{g}$. Then the transverse complex structure on M can be studied in terms of an ordinary complex structure on $M \times G$. Recall that a manifold S is Sasakian if its Riemannian cone $S \times \mathbb{R}$ is Kähler – and thus the above approach is a starting point for natural generalisations.

We point out that an integrable complex structure on its Lie algebra does not turn a Lie group into a complex Lie group. In fact, every compact Lie group of even dimension admits an integrable left invariant complex structure, cf. [8,10], while it is well-known that only tori can be compact complex Lie groups.

In the last section we provide an explicit integrable complex structure for every algebra of the type $\mathfrak{o}(n) \times \mathfrak{o}(n)$.

2. Complex structures on 6-dimensional product algebras

Recall that the 3-dimensional Lie algebras were classified into 9 types by Bianchi. We use a variant, a classification from [6] by the dimension of the derived algebra and Jordan decomposition of certain automorphism acting upon it. We include the statement for convenience.

Proposition 1. *[6] Let e_1, e_2 and e_3 be a basis of $\mathbb{R}^3$. Up to isomorphism of Lie algebras, the following list yields all Lie brackets on $\mathbb{R}^3$*

(1) $[e_1, e_3] = 0$, $[e_2, e_3] = 0$, $[e_1, e_2] = 0$
(2) $[e_1, e_3] = 0$, $[e_2, e_3] = 0$, $[e_1, e_2] = e_1$
(3) $[e_1, e_3] = 0$, $[e_2, e_3] = 0$, $[e_1, e_2] = e_3$
(4) $[e_1, e_3] = e_1$, $[e_2, e_3] = \theta e_2$, $[e_1, e_2] = 0$ *for* $\theta \neq 0$ *(the case* $\theta = 1$ *is considered to be Bianchi's ninth type)*
(5) $[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$, $[e_1, e_2] = 0$
(6) $[e_1, e_3] = \theta e_1 - e_2$, $[e_2, e_3] = e_1 + \theta e_2$, $[e_1, e_2] = 0$ *for* $\theta \neq 0$
(7) $[e_1, e_3] = e_2$, $[e_2, e_3] = e_1$, $[e_1, e_2] = e_3$
(8) $[e_1, e_3] = -e_2$, $[e_2, e_3] = e_1$, $[e_1, e_2] = e_3$

We fix some notation. Whenever we write $(x, y, z) \in \mathfrak{g}$, it is understood in the appropriate basis above. If any other basis $\{u, v, w\}$ is used, we write $Xu + Yv + Zw$.

The direct product $\mathfrak{g} \times \mathfrak{g}$ inherits the bracket operation on each factor from $\mathfrak{g}$: $[(u, v), (t, s)] = ([u, t], [v, s])$. We keep the distinction between the two copies of $\mathfrak{g}$ inside $\mathfrak{g} \times \mathfrak{g}$ by adding asterisks to the second copy. Any vector decorated with an asterisk is understood to lie in $\mathfrak{g}^* = 0 \times \mathfrak{g}$, while those without it lie in $\mathfrak{g} = \mathfrak{g} \times 0$. We tacitly use the natural isomorphism between the two copies, $(v, 0)^* = (0, v)$. We also distinguish the two components of a complex structure $\mathcal{J}$ – it will be convenient to work with J and J^* as in $\mathcal{J}v = (Jv, J^*v)$ to indicate its $\mathfrak{g}$- and $\mathfrak{g}^*$-parts separately. As a rule, virtually every vector on which we act in the lengthy proofs lies in $\mathfrak{g}$.

To finish the preliminaries we note the following to use frequently in what follows.

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