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Centers of generalized quantum groups

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A B S T R A C T

This paper treats the generalized quantum group $U = U(\chi, \pi)$ with a bi-homomorphism χ for which the corresponding generalized root system is a finite set. We establish a Harish-Chandra type theorem describing the (skew) centers of U.

1. Introduction

In this paper, we study the center and skew-centers of the generalized quantum groups. To do so, we use some facts of Zariski topology. We introduce a sub-topology of it, and call it *Nichols topology*. Study of the topology has virtually been initiated by [11]. The main result of this paper is Theorem 10.4. As a bi-product, we also give its 'symmetric' version, Theorem 1.2. Theorem 7.5 is just [11, Theorem 7.3]. Theorem 7.3 and Theorem 4.6 are similar to [11, Lemma 6.7] and [11, Lemma 9.4] respectively. We follow the Kac argument [12] for the center. However the proof of Theorem 8.8 is very delicate since we treat 'all roots of unit cases.' The proofs of Proposition 9.5 and Lemmas 9.2 and 10.2 are original. We adopt the new and simpler definition [24] for the generalized root systems (see (5.1)), and in Section 5, we clarify relation between the new and conventional ones. Argument in Subsection 8.1 has originally been given in [11, Section 7]. We need delicate arguments in Section 8 to obtain Theorem 8.8.

Let $\mathbb{K}$ be an algebraically closed field. Let $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$. Let I be a finite set. Let $\mathfrak{A}$ be a free $\mathbb{Z}$-module with $|I| = \mathrm{rank}_\mathbb{Z}\mathfrak{A}$. Let $\chi : \mathfrak{A} \times \mathfrak{A} \to \mathbb{K}^\times$ be a bi-homomorphism, that is,

$$\chi(\lambda, \mu + \nu) = \chi(\lambda, \mu)\chi(\lambda, \nu) \quad \text{and} \quad \chi(\lambda + \mu, \nu) = \chi(\lambda, \nu)\chi(\mu, \nu) \tag{1.1}$$

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for all $\lambda,\ \mu,\ \nu \in \mathfrak{A}$. In Introduction, for simplicity, we also assume $\chi(\lambda,\mu)=\chi(\mu,\lambda)$ for all $\lambda,\ \mu \in \mathfrak{A}$. Let $\pi : I \to \mathfrak{A}$ be a map such that $\pi(I)$ is a $\mathbb{Z}$-base of $\mathfrak{A}$. In the same way as in the Lusztig's definition [16] of the quantum groups, to the pair (χ,π), we can associate a unique $\mathbb{K}$-algebra $\check{U}(\chi,\pi)$ characterized by the following properties.

(i) As a $\mathbb{K}$-algebra, $\check{U}$ has generators $\check{K}_\lambda$ $(\lambda \in \mathfrak{A})$, $\check{E}_i$, $\check{F}_i$ $(i \in I)$ satisfying the equations $\check{K}_0 = 1$, $\check{K}_\lambda\check{K}_\mu = \check{K}_{\lambda+\mu}$, $\check{K}_\lambda\check{E}_i = \chi(\lambda,\pi(i))\check{E}_i\check{K}_\lambda$, $\check{K}_\lambda\check{F}_i = \chi(\lambda,-\pi(i))\check{F}_i\check{K}_\lambda$, $\check{E}_i\check{F}_j - \check{F}_j\check{E}_i = \delta_{ij}(-\check{K}_{\pi(i)} + \check{K}_{-\pi(i)})$. As a $\mathbb{K}$-linear space, the elements $\check{K}_\lambda$ $(\lambda\in\mathfrak{A})$ are linearly independent.

(ii) Let $\check{U}^0$ be the subalgebra of $\check{U}$ generated by $\check{K}_\lambda$ $(\lambda \in \mathfrak{A})$. Let $\check{U}^+$ (resp. $\check{U}^-$) be the subalgebra of $\check{U}$ generated by $\check{E}_i$ (resp. $\check{F}_i$) $(i \in I)$ and 1. One has a $\mathbb{K}$-linear isomorphism $\check{U}^- \otimes_{\mathbb{K}} \check{U}^0 \otimes_{\mathbb{K}} \check{U}^+ \to \check{U}$ with $Y \otimes Z \otimes X \mapsto YZX$. One also has $\mathbb{K}$-linear subspaces $\check{U}_\lambda$ $(\lambda \in \mathfrak{A})$ such that $\check{U} = \oplus_{\lambda\in\mathfrak{A}}\check{U}_\lambda$, $\check{U}_\lambda\check{U}_\mu \subset \check{U}_{\lambda+\mu}$, $\check{U}^0 \subset \check{U}_0$, and $\check{E}_i \in \check{U}_{\pi(i)}$, $\check{F}_i \in \check{U}_{-\pi(i)}$ $(i \in I)$.

(iii) Let $\check{U}^{\geq}$ (resp. $\check{U}^{\leq}$) be the subalgebra of $\check{U}$ generated by $\check{U}^+$ (resp. $\check{U}^-$) and $\check{U}^0$. One has a Drinfeld bilinear map $\check{\vartheta} = \check{\vartheta}^{\chi,\pi} : \check{U}^{\geq} \times \check{U}^{\leq} \to \mathbb{K}$ such that $\check{\vartheta}_{|\check{U}^+\times\check{U}^-} : \check{U}^+ \times \check{U}^- \to \mathbb{K}$ is non-degenerate, and $\check{\vartheta}(\check{K}_\lambda,\check{K}_\mu) = \chi(\lambda,-\mu)$, $\check{\vartheta}(\check{E}_i,\check{F}_j) = \delta_{ij}$, $\check{\vartheta}(\check{E}_i,\check{K}_\mu) = \check{\vartheta}(\check{K}_\lambda,\check{F}_j) = 0$.

We note that for each pair (χ,π), one define the Karchenko's root system $R(\chi,\pi)$, which is a subset of $\mathfrak{A}$ including $\pi(I)$; $R(\chi,\pi)$ is used to give a PBW-type theorem of $\check{U}(\chi,\pi)$, see Theorem 6.4.

Example 1.1. Assume that $\mathbb{K}$ is the complex field $\mathbb{C}$. Let $q \in \mathbb{C}^\times \setminus \{1\}$. Let $\langle\,,\,\rangle : \mathfrak{A}\times\mathfrak{A} \to \mathbb{Z}$ be a $\mathbb{Z}$-module bihomomorphism.

(1) Let $\mathfrak{g}$ be a complex simple Lie algebra. Assume that $\langle\pi(i),\pi(i)\rangle \neq 0$ $(i \in I)$ and $\left[\frac{2\langle\pi(i),\pi(j)\rangle}{\langle\pi(i),\pi(i)\rangle}\right]_{i,j\in I}$ coincides with the Cartan matrix of $\mathfrak{g}$. Assume that q is not a root of unity, and that $\chi(\pi(i),\pi(j)) = q^{\langle\pi(i),\pi(j)\rangle}$ $(i,\ j \in I)$. Then $\check{U}$ can be identified with the quantum group $U_q(\mathfrak{g})$, and $R(\chi,\pi)$ can be identified with the root system of $\mathfrak{g}$.

(2) Assume that $\langle\,,\,\rangle$ is as in (1). Assume that q is an n-th primitive root of unity for some $n \in \mathbb{N}$ with $n \geq 2$. Assume that $\chi(\pi(i),\pi(j)) = q^{\langle\pi(i),\pi(j)\rangle}$ $(i,\ j \in I)$, $\chi(\pi(i),\pi(i)) \neq 1$ $(i \in I)$ and $\chi(\pi(i),\pi(i)) \neq \chi(\pi(j),\pi(j))$ $(i,\ j \in I$ with $\langle\pi(i),\pi(i)\rangle \neq \langle\pi(j),\pi(j)\rangle)$. Let X be the two-sided ideal of $\check{U}$ generated by $\check{K}^n_{\pi(i)} - 1$ $(i \in I)$. Then the quotient algebra $\check{U}/X$ can be identified with the Lusztig's small quantum group $u_q(\mathfrak{g})$. It follows that $R(\chi,\pi)$ is the same as in (1).

(3) (See also Subsection 11.2.) Let $\mathfrak{s}$ be a basic classical Lie superalgebra (which is also called a complex simple Lie superalgebra of type A–G). Identify $\mathfrak{A}$ with the root lattice of $\mathfrak{s}$ for which $\pi(I)$ is also a set of simple roots. Identify $\langle\,,\,\rangle$ with the Killing form of $\mathfrak{s}$. Let $I' := \{i \in I|\ \pi(i)$ is an odd simple root$\}$. Define the $\mathbb{Z}$-module homomorphism $\partial : \mathfrak{A} \to \mathbb{Z}$ by $\partial(\pi(i)) := 0$ $(i \in I\setminus I')$ and $\partial(\pi(j)) := 1$ $(j \in I')$. Assume that q is not a root of unity, and that $\chi(\pi(i),\pi(j)) = (-1)^{\partial(\pi(i))\partial(\pi(j))}q^{\langle\pi(i),\pi(j)\rangle}$ $(i,\ j \in I)$. Let $\check{U}^\sigma = \check{U} \oplus \check{U}\sigma$ be the $\mathbb{C}$-algebra obtained from $\check{U}$ by adding an element σ such that $\sigma^2 = 1$, $\sigma\check{K}_\lambda\sigma = \check{K}_\lambda$, $\sigma\check{E}_i\sigma = (-1)^{\partial(\pi(i))}\check{E}_i$, and $\sigma\check{F}_i\sigma = (-1)^{\partial(\pi(i))}\check{F}_i$ $(i \in I)$. Then the quantum superalgebra $U_q(\mathfrak{s})$ can be identified with the subalgebra of $\check{U}^\sigma$ generated by $\sigma^{\partial(\pi(i))}\check{K}_{\pi(i)}$, $\check{E}_i$, and $\check{F}_i\sigma^{\partial(\pi(i))}$, so $\check{U}^\sigma = U_q(\mathfrak{s}) \oplus U_q(\mathfrak{s})\sigma$. It follows that the root system of $\mathfrak{s}$ is given by $R(\chi,\pi) \cup \{2\beta|\beta \in R(\chi,\pi), (-1)^{\partial(\beta)} = -1, \langle\beta,\beta\rangle \neq 0\}$.

Let $\omega : \mathfrak{A} \to \mathbb{K}^\times$ be a $\mathbb{Z}$-module homomorphism. We call the $\mathbb{K}$-linear space

$$\check{\mathfrak{Z}}_\omega = \check{\mathfrak{Z}}_\omega(\chi,\pi) := \{\, Z \in \check{U}_0 \,|\, \forall\lambda \in \mathfrak{A},\ \forall X \in \check{U}_\lambda,\ ZX = \omega(\lambda)XZ \,\}$$

the *ω-skew graded center* of $\check{U}$. Define the $\mathbb{K}$-linear map $\check{\mathfrak{Sh}} = \check{\mathfrak{Sh}}^{\chi,\pi} : \check{U} \to \check{U}^0$ by $\check{\mathfrak{Sh}}_{|\check{U}^0} = \mathrm{id}_{\check{U}^0}$ and $\check{\mathfrak{Sh}}(\check{U}\check{E}_i) = \check{\mathfrak{Sh}}(\check{F}_i\check{U}) = \{0\}$ $(i \in I)$. Let $\check{\mathfrak{HC}}_\omega = \check{\mathfrak{HC}}^{\chi,\pi}_\omega := \check{\mathfrak{Sh}}_{|\check{\mathfrak{Z}}_\omega}$. It is easy to see that $\check{\mathfrak{HC}}_\omega$ is injective.

The $\check{U}$ is also appeared in the context of the Schneider–Andruskiewitsch program of classification of pointed Hopf algebras (see [1]). Let $R := R(\chi,\pi)$. Heckenberger classified R with $\mathrm{Char}(\mathbb{K}) = 0$ and $|R| < \infty$ (see [8]). In 2010, Heckenberger and the second author [11] gave a factorization formula of the Shapovalov