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Two characterisations of groups amongst monoids

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A B S T R A C T

The aim of this paper is to solve a problem proposed by Dominique Bourn: to provide a categorical-algebraic characterisation of groups amongst monoids and of rings amongst semirings. In the case of monoids, our solution is given by the following equivalent conditions:

(i) G is a group;
(ii) G is a *Mal'tsev object*, i.e., the category $\mathsf{Pt}_G(\mathsf{Mon})$ of points over G in the category of monoids is unital;
(iii) G is a *protomodular object*, i.e., all points over G are *stably strong*, which means that any pullback of such a point along a morphism of monoids $Y \to G$ determines a split extension

$$0 \longrightarrow K \overset{k}{\rightarrowtail} X \underset{f}{\overset{s}{\leftrightarrows}} Y \longrightarrow 0$$

in which k and s are jointly strongly epimorphic.

We similarly characterise rings in the category of semirings.

On the way we develop a *local* or *object-wise* approach to certain important conditions occurring in categorical algebra. This leads to a basic theory involving what we call *unital* and *strongly unital* objects, *subtractive* objects, *Mal'tsev* objects and *protomodular* objects. We explore some of the connections between these new notions and give examples and counterexamples.

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1. Introduction

The concept of *abelian object* plays a key role in categorical algebra. In the study of categories of non-abelian algebraic structures—such as groups, Lie algebras, loops, rings, crossed modules, etc.—the "abelian case" is usually seen as a basic starting point, often simpler than the general case, or sometimes even trivial. Most likely there are known results which may or may not be extended to the surrounding non-abelian setting. Part of categorical algebra deals with such generalisation issues, which tend to become more interesting precisely where this extension is not straightforward. Abstract commutator theory for instance, which is about *measuring non-abelianness*, would not exist without a formal interplay between the abelian and the non-abelian worlds, enabled by an accurate definition of abelianness.

Depending on the context, several approaches to such a conceptualisation exist. Relevant to us are those considered in [3]; see also [25,35,33] and the references in [3]. The easiest is probably to say that an **abelian object** is an object which admits an internal abelian group structure. This makes sense as soon as the surrounding category is *unital*—a condition introduced in [5], see below for details—which is a rather weak additional requirement on a pointed category implying that an object admits at most one internal abelian group structure. So that, in this context, "being abelian" becomes a property of the object in question.

The full subcategory of a unital category $\mathbb{C}$ determined by the abelian objects is denoted $\mathsf{Ab}(\mathbb{C})$ and called the **additive core** of $\mathbb{C}$. The category $\mathsf{Ab}(\mathbb{C})$ is indeed additive, and if $\mathbb{C}$ is a finitely cocomplete regular [2] unital category, then $\mathsf{Ab}(\mathbb{C})$ is a reflective [3] subcategory of $\mathbb{C}$. If $\mathbb{C}$ is moreover Barr exact [2], then $\mathsf{Ab}(\mathbb{C})$ is an abelian category, and called the **abelian core** of $\mathbb{C}$.

For instance, in the category Lie_K of Lie algebras over a field K, the abelian objects are K-vector spaces, equipped with a trivial (zero) bracket; in the category Gp of groups, the abelian objects are the abelian groups, so that $\mathsf{Ab}(\mathsf{Gp}) = \mathsf{Ab}$; in the category Mon of monoids, the abelian objects are abelian groups as well: $\mathsf{Ab}(\mathsf{Mon}) = \mathsf{Ab}$; etc. In all cases the resulting commutator theory behaves as expected.

1.1. Beyond abelianness: weaker conditions

The concept of an abelian object has been well studied and understood. For certain applications, however, it is too strong: the "abelian case" may not just be *simple*, it may be *too simple*. Furthermore, abelianness may "happen too easily". As explained in [3], the Eckmann–Hilton argument implies that any internal monoid in a unital category is automatically a *commutative* object. For instance, in the category of monoids any internal monoid is commutative, so that in particular an internal group is always abelian: $\mathsf{Gp}(\mathsf{Mon}) = \mathsf{Ab}$. Amongst other things, this fact is well known to account for the abelianness of the higher homotopy groups.

If we want to capture groups amongst monoids, avoiding abelianness turns out to be especially difficult. One possibility would be to consider gregarious objects [3], because the "equation"

$$\text{commutative} + \text{gregarious} = \text{abelian}$$

holds in any unital category. But this notion happens to be too weak, since examples were found of gregarious monoids which are not groups. On the other hand, as explained above, the concept of an internal group is too strong, since it gives us abelian groups. Whence the subject of our present paper: to find out how to

characterise *non-abelian* groups inside the category of monoids

in categorical-algebraic terms. That is to say, is there some weaker concept than that of an abelian object which, when considered in Mon, gives the category Gp?

This question took quite a long time to be answered. As explained in [14,15], the study of monoid actions, where an **action** of a monoid B on a monoid X is a monoid homomorphism $B \to \mathrm{End}(X)$ from B to the

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