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Nested Artin strong approximation property

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A B S T R A C T

We study the Artin Approximation property with constraints in a different frame. As a consequence we give a nested Artin Strong Approximation property for algebraic power series rings over a field.

0. Introduction

Let K be a field and $R = K\langle x\rangle$, $x = (x_1, \dots, x_n)$ be the ring of algebraic power series in x over K, that is the algebraic closure of the polynomial ring $K[x]$ in the formal power series ring $\hat{R} = K[[x]]$. Let $f = (f_1, \dots, f_q)$ be a system of polynomials in $Y = (Y_1, \dots, Y_p)$ over R and $\hat{y}$ be a solution of f in the completion $\hat{R}$ of R.

Theorem 1 *(M. Artin [2]). For any $c \in \mathbf{N}$ there exists a solution $y^{(c)}$ in R such that $y^{(c)} \equiv \hat{y}$ mod $(x)^c$.*

Also M. Artin proved before (see [1]) a similar statement for the ring R of complex convergent power series and later (see [3, p. 7]) asked, whether, given $c \in \mathbb{N}$ and a formal solution $y(x) = (y_1(x), \dots, y_p(x)) \in \mathbb{C}[[x]]^p$ satisfying

$$y_j(x) \in K[[x_1, \dots, x_{s_j}]] \quad \forall j \in [p]$$

for some integers $s_j \in [n]$, there exists a convergent solution $y'(x)$ of f such that $y'(x) \equiv y(x)$ mod $(x)^c$ and

$$y'_j(x) \in k\{x_1, \dots, x_{s_j}\} \quad \forall j \in [p].$$

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Shortly after, A. Gabrielov [8] (see also [20, Example 5.3.1]) gave an example showing that the answer to the previous question is negative in general. On the other hand, since Theorem 1 remains valid if we replace convergent power series by algebraic power series, the question of M. Artin is also relevant in this context and it appears that in this case the question has a positive answer as it is shown in [17, Theorem 3.7] (see also [18, Corollary 3.7], [20, Theorem 5.2.1]).

Question 2. *(Artin Approximation with constraints [20, Problem 1, page 65]) Let R be an excellent local subring of $K[[x]]$, $x = (x_1, \dots, x_n)$ such that the completion of R is $K[[x]]$ and $f \in R[Y]^q$, $Y = (Y_1, \dots, Y_p)$. Assume that there exists a formal solution $\hat{y} \in K[[x]]^p$ of $f = 0$ such that $\hat{y}_i \in K[[\{x_j : j \in J_i\}]]$ for some subset $J_i \subset [n]$, $i \in [p]$. Is it possible to approximate $\hat{y}$ by a solution $y \in R^p$ of $f = 0$ such that $y_i \in R \cap K[[\{x_j : j \in J_i\}]]$, $i \in [p]$?*

Similarly, we considered below the following question.

Question 3. *Let $K \subset K'$ be a field extension, $R = K[x]_{(x)}$, $R' = K'[x]_{(x)}$ (resp. $R = K\langle x\rangle$, $R' = K'\langle x\rangle$) and $f \in R[Y]^p$, $Y = (Y_1, \dots, Y_m)$. Assume that there exists a solution $\hat{y} \in R'^p$ of $f = 0$ such that $\hat{y}_i \in K'[\{x_j : j \in J_i\}]_{(x_{J_i})}$ (resp. $K'\langle\{x_j : j \in J_i\}\rangle$) for some subset $J_i \subset [n]$, $i \in [m]$. Is it possible to find a solution $y \in R^p$ of $f = 0$ such that $y_i \in K[\{x_j : j \in J_i\}]_{(x_{J_i})}$ (resp, $K\langle\{x_j : j \in J_i\}\rangle$), and $\operatorname{ord} y_i = \operatorname{ord} \hat{y}_i$, $i \in [p]$?*

We show (see Proposition 7 and Theorem 9) that Question 3 has a positive answer when the field extension $K \subset K'$ is algebraically pure. A ring morphism $u : A \to B$ is called *algebraically pure* (see [16]), if every finite system of polynomial equations over A has a solution in A if it has a solution in B. A finite type ring morphism is algebraically pure if and only if it has a retraction and a filtered inductive limit of algebraically pure morphisms is an algebraically pure morphism by [16] (see also [14, Theorem 1.10]). It is easy to see that a field extension of an algebraically closed field is algebraically pure and the ultrapower of fields define algebraically pure field extension (see the proof of Theorem 11).

The above questions are related with the so called Artin approximation property. There exists also a strong approximation property (see [10,2,15,11,16–18,20,19]).

Question 4. *(Strong Artin Approximation with constraints [20, Problem 2, page 65]) Let us consider $f(y) \in K[[x]][y]^q$ and $J_i \subset [n]$, $i \in [p]$. Does there exist a function $\nu : \mathbb{N} \to \mathbb{N}$ such that for all $c \in \mathbb{N}$ and all $\hat{y}_i(x) \in K[[x_{J_i}]]$, $j \in J_i$, $i \in [p]$, such that $f(\hat{y}(x)) \in (x)^{\nu(c)}$, $\hat{y}(x) = (\hat{y}_1(x), \dots, \hat{y}_p(x))$, there exist $y_i(x) \in K[[x_{J_i}]]$, $i \in [m]$ such that $f(y(x)) = 0$, $y(x) = (y_1(x), \dots, y_p(x))$ and $\hat{y}_i(x) \equiv y_i(x)$ mod $(x)^c$, $i \in [p]$?*

This question has a positive answer when $K = \mathbb{C}$ but a negative one when $K = \mathbb{Q}$ (see [4], [20, Proposition 3.3.4] and [20, Example 5.4.5])). Similarly, we considered below the following nested question.

Question 5. *Let us consider a field K, $f = (f_1, \dots, f_q) \in K\langle x\rangle[Y]^q$, $Y = (Y_1, \dots, Y_p)$ and $0 \le s_1 \le \dots \le s_p \le n$ be some non-negative integers. Does there exist a function $\nu : \mathbb{N}^p \to \mathbb{N}$ such that for all $c \in \mathbb{N}^p$ and all $\hat{y}_i(x) \in K[x_1, \dots, x_{s_i}]$, such that $\operatorname{ord} \hat{y}_i(x) = c_i$, $i \in [p]$ and $f(\hat{y}(x)) \in (x)^{\nu(c)}$, $\hat{y}(x) = (\hat{y}_1(x) \dots, \hat{y}_p(x))$, there exist $y_i(x) \in K\langle x_1, \dots, x_{s_i}\rangle$, $i \in [p]$ such that $f(y(x)) = 0$, $y(x) = (y_1(x), \dots, y_p(x))$ and $\operatorname{ord} y_i(x) = c_i$, $i \in [p]$?*

Theorem 11 shows a positive answer to this question. The proof uses the ultrapower methods (see [4, 16–18]) and so it is not constructive. We should mention that there exists a stronger result (see Remark 13).

Artin approximation property with constraints is necessary in CR Geometry (see [5] and [12]), the nested case appears in the construction of the analytic deformations of a complex analytic germ when it has an