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On the derived category of quasi-coherent sheaves on an Adams geometric stack ☆

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Let $\mathbf{X}$ be an Adams geometric stack. We show that $\mathbf{D}(\mathsf{A}_{\mathsf{qc}}(\mathbf{X}))$, its derived category of quasi-coherent sheaves, satisfies the axioms of a stable homotopy category defined by Hovey, Palmieri and Strickland in [13]. Moreover we show how this structure relates to the derived category of comodules over a Hopf algebroid that determines $\mathbf{X}$.

1. Introduction

Let X be a quasi-compact and semi-separated scheme. In [1] it is shown that its derived category of quasi-coherent sheaves $\mathbf{D}(\mathsf{A}_{\mathsf{qc}}(X))$ satisfies the axioms of a stable homotopy category from [13]. During the preparation of that paper we were asked whether the same result holds for the derived category of quasi-coherent sheaves on an algebraic stack. Unfortunately, the available references [21] and [24] suffered from some inaccuracies, and most importantly, did not contain the existence of generators in the category of quasi-coherent sheaves. To remedy this we embarked on a project of reconciling all the available definitions and settling the question of existence of generators. The project has born fruit in the form of [2].

In the latter paper we stick to the context of *geometric stacks*, *i.e.* those that are quasi-compact and possess an affine diagonal (in other words, they are semi-separated). This is a minimal requirement for our goal in view of the necessity of this hypothesis already in the scheme case. On the other hand, these

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stacks admit a representation by an *affine* groupoid scheme, which corresponds by the algebra–geometry duality to an algebraic gadget called a Hopf algebroid. These objects play an important role in homotopy theory, in the context of orientable generalized cohomology theories and the main reference for this is [25]. In this setting, quasi-coherent sheaves on a geometric stack correspond to comodules over the defining Hopf algebroid. This was used in a crucial way in [7] and [12] where the authors looked at the moduli stack of formal groups which is an *ind*-geometric stack. In general, the main properties of quasi-coherent sheaves using comodules were studied by Hovey in [15] and [16].

In this paper, we will follow the general conventions and notations in [21] and [2]. Our setup differs from the one in [29]; this paper is essentially independent of it. Their treatment is more general due to the fact that their algebraic stacks have less strict conditions on their diagonals.

Let us explain some differences between the present approach and the one at [29]. A *big site* over a scheme is defined by assigning a covering condition on a certain collection of schemes, but no further condition on the schemes that constitute the site. This collection should be a set but big enough to be closed for certain set-theoretic operations. In [4] the notion of universes is used, but some constructions depend on the chosen underlying universe. In Stacks Project, there is a choice of a partial universe that defines a small category of schemes. The usual set-theoretic operations are possible in this context as follows from [29].

In [29], the setting is of big flat sites relative to this choice. This has the merit of making the functoriality of the corresponding categories of sheaves of modules and its derived counterparts almost automatic. A *small site* over a scheme is defined by a property that defines the coverings and also determines the schemes of the site. In particular, the underlying set is not a partial universe. The functoriality of the corresponding topos does not hold if the topology is finer than the étale one, like the topology used in this paper. For quasi-coherent sheaves the functoriality properties can be restored though, as it is explained in [2].

The category $\mathcal{O}_{\mathsf{X}}\text{-}\mathsf{Mod}$ considered in [29] depends on the choice of the partial universe, because the size of the modules of sections over the objects of the site is bounded by the bigger cardinal available. By [29], the category $\mathbf{D}_{\mathsf{qc}}(\mathsf{Sch}_{\mathsf{fppf}}/\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ does not depend on this choice because it is equivalent to $\mathbf{D}_{\mathsf{qc}}(\mathbf{X}_{\mathsf{liss-\acute{e}t}}, \mathcal{O}_{\mathbf{X}})$ and $\mathbf{X}_{\mathsf{liss-\acute{e}t}}$ does not change with the partial universe. In this paper, $\mathbf{D}_{\mathsf{qc}}(\mathbf{X}) := \mathbf{D}_{\mathsf{qc}}(\mathbf{X}_{\mathsf{fppf}}, \mathcal{O}_{\mathbf{X}})$ where $\mathbf{X}_{\mathsf{fppf}}$ denotes the small flat site. This category agrees with $\mathbf{D}_{\mathsf{qc}}(\mathsf{Sch}_{\mathsf{fppf}}/\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ because quasi-coherent sheaves are the same on $\mathbf{X}_{\mathsf{fppf}}$ and $\mathbf{X}_{\mathsf{liss-\acute{e}t}}$ as they both correspond to Cartesian presheaves, see [2, Theorem 3.12].

We have been asked about the choice of $\mathbf{D}(\mathsf{A}_{\mathsf{qc}}(\mathbf{X}))$ instead of the usual $\mathbf{D}_{\mathsf{qc}}(\mathbf{X})$. A simple reason is that under our hypothesis there is an equivalence between $\mathbf{D}^+(\mathsf{A}_{\mathsf{qc}}(\mathbf{X}))$ and $\mathbf{D}^+_{\mathsf{qc}}(\mathbf{X})$ (Proposition 2.6). Notice that without the semi-separation hypothesis this equivalence need not hold.

It turns out that the existence of nice generators in the category of quasi-coherent sheaves is problematic. We have to impose the so called *Adams condition*. This property of the category of comodules over a Hopf algebroid is equivalent to the classical resolution property on schemes (see the discussion in section 3). Under this additional hypothesis, one can prove the existence of dualizable generators and a structure of symmetric closed monoidal category on the derived category, thus fulfilling the axioms of Hovey, Palmieri and Strickland. Unlike the case of schemes, on geometric stacks the existence of *compact* generators or, more precisely, the question whether dualizable complexes are compact is a delicate one. The failure is due to the existence of stacks with infinite homological dimension, as the classifying stack of an algebraic group attests. In the case of finite homological dimension, Hall and Rydh proved that this difficulty does not arise [10]. One may think that cohomology of quasi-coherent sheaves on an algebraic stack generalizes both cohomology of quasi-coherent sheaves on a scheme and cohomology of representations on a group scheme.

In a sense this paper addresses the same problem as Hovey's [16]. The results are of a different sort. In Hovey's words, he considers the stable *homotopy* theory of comodules rather than its *homology*. In practice, this means that he considers an a priori different localization of the categories of complexes of quasi-coherent sheaves on a stack. We ignore if both categories agree but see 6.11 for a detailed discussion. In any case, we stress that our methods differ from those in [16], as there homotopical algebra and model categories are used while in the present paper we employ just homological algebra and derived categories.

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