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On integrally closed simple extensions of valuation rings ☆

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A B S T R A C T

Let v be a Krull valuation of a field with valuation ring R_v. Let θ be a root of an irreducible trinomial $F(x) = x^n + ax^m + b$ belonging to $R_v[x]$. In this paper, we give necessary and sufficient conditions involving only a, b, m, n for $R_v[\theta]$ to be integrally closed. In the particular case when v is the p-adic valuation of the field $\mathbb{Q}$ of rational numbers, $F(x) \in \mathbb{Z}[x]$ and $K = \mathbb{Q}(\theta)$, then it is shown that these conditions lead to the characterization of primes which divide the index of the subgroup $\mathbb{Z}[\theta]$ in A_K, where A_K is the ring of algebraic integers of K. As an application, it is deduced that for any algebraic number field K and any quadratic field L not contained in K, we have $A_{KL} = A_K A_L$ if and only if the discriminants of K and L are coprime.

1. Introduction

Let R be an integrally closed domain and θ be an element of an integral domain containing R with θ integral over R. The question "when is $R[\theta]$ integrally closed" has inspired many mathematicians (cf. [1,8,9,13]). This problem is closely related with the existence of a power basis of an algebraic number field. Recall that a power basis of an algebraic number field K is a $\mathbb{Z}$-basis of the ring of algebraic integers of K consisting of powers of a single element; indeed θ would be such an element if and only if $\mathbb{Z}[\theta]$ is integrally closed in its quotient field K. If A_K denotes the ring of algebraic integers of an algebraic number field $K = \mathbb{Q}(\theta)$ with θ an algebraic integer and $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at a nonzero prime ideal $p\mathbb{Z}$, then using Lagrange's theorem and Cauchy's theorem for finite groups, it can be easily seen that a prime p does not divide $[A_K : \mathbb{Z}[\theta]]$ if and only if $A_K \subseteq \mathbb{Z}_{(p)}[\theta]$ which is the same as saying that $\mathbb{Z}_{(p)}[\theta]$ is integrally closed. In 1878, Dedekind gave a necessary and sufficient criterion to be satisfied by the minimal polynomial $F(x)$ of θ over $\mathbb{Q}$ so that $p \nmid [A_K : \mathbb{Z}[\theta]]$. He proved that if $\overline{F}(x) = \bar{g}_1(x)^{e_1} \cdots \bar{g}_t(x)^{e_t}$ is the factorization of

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the polynomial $\overline{F}(x)$ obtained by replacing coefficients of $F(x)$ modulo p as a product of powers of distinct irreducible polynomials over $\mathbb{Z}/p\mathbb{Z}$ with $g_i(x)$ monic, then $\mathbb{Z}_{(p)}[\theta]$ is integrally closed if and only if for each i, either $e_i = 1$ or $\bar{g}_i(x) \nmid \overline{M}(x)$, where $M(x) = \frac{1}{p}(F(x) - \prod_{i=1}^{t} g_i(x)^{e_i})$ (see [2, Theorem 6.1.4], [3]). As $\mathbb{Z}_{(p)}$ is the valuation ring of the p-adic valuation of rationals, the above criterion gives a motivation to investigate when is a simple ring extension of a valuation ring integrally closed (see [4] for valuations). In 2006, Ershov (cf. [5,9]) extended this criterion to arbitrary valuation rings and proved the following:

Theorem 1.A *(Generalized Dedekind Criterion). Let v be a Krull valuation of arbitrary rank of a field with valuation ring R_v having maximal ideal M_v. For $g(x) \in R_v[x]$, let $\bar{g}(x)$ denote the polynomial obtained on replacing each coefficient of $g(x)$ by its image under the canonical homomorphism from R_v onto R_v/M_v. Let $F(x) \in R_v[x]$ be a monic irreducible polynomial having a root θ in its splitting field and $\overline{F}(x) = \bar{g}_1(x)^{e_1} \cdots \bar{g}_t(x)^{e_t}$ be the factorization of $\overline{F}(x)$ into a product of powers of distinct irreducible polynomials over R_v/M_v with $g_i(x) \in R_v[x]$ monic. Then $R_v[\theta]$ is integrally closed if and only if either $e_i = 1$ for each i or some $e_j > 1$, in which case M_v is a principal ideal say generated by π and $\bar{g}_j(x)$ does not divide $\overline{M}(x)$ for such an index j, where $M(x) = \frac{1}{\pi}(F(x) - g_1(x)^{e_1} \cdots g_t(x)^{e_t})$.*

In this paper, we use the above theorem to give necessary and sufficient conditions involving a, b, m, n for $R_v[\theta]$ to be integrally closed when θ is a root of an irreducible trinomial[1] $F(x) = x^n + ax^m + b$ belonging to $R_v[x]$. In what follows, v, R_v, M_v are as in Theorem 1.A. For an element α belonging to R_v, $\bar{\alpha}$ will denote its image under the canonical homomorphism from R_v onto R_v/M_v. When a polynomial $g(x)$ belongs to $R_v[x]$, $\bar{g}(x)$ will have the same meaning as in Theorem 1.A. We shall denote by D the discriminant of the trinomial $F(x) = x^n + ax^m + b$. It is known (cf. [12]) that

$$D = (-1)^{\binom{n}{2}} b^{m-1}[b^{n_1-m_1}n^{n_1} - (-1)^{n_1}a^{n_1}m^{m_1}(n-m)^{n_1-m_1}]^{d_0} \tag{1}$$

where $d_0 = gcd(m, n)$, $n_1 = \frac{n}{d_0}$, $m_1 = \frac{m}{d_0}$. In this paper, we prove

Theorem 1.1. *Let v be a Krull valuation of arbitrary rank of a field having valuation ring R_v, maximal ideal M_v and perfect residue field. Let p denote the characteristic of the residue field R_v/M_v in case it is positive. Let θ be a root of a monic irreducible trinomial $F(x) = x^n + ax^m + b$ belonging to $R_v[x]$ and d_0, m_1, n_1, D be as above. Assume[2] that $v(D) > 0$. Then $R_v[\theta]$ is integrally closed if and only if M_v is a principal ideal say generated by π and one of the following conditions is satisfied:*

(i) when a, b belong to M_v, then $v(b) = v(\pi)$;
(ii) when $a \in M_v$ and $b \notin M_v$ with $j \geq 1$ as the highest power of p dividing n, then either $v(a_2) \geq v(\pi)$ and $v(b_1) = 0$ or $v(a_2) = 0 = v((-b)^{m_1}a_2^{n_1} - (-b_1)^{n_1})$, where $a_2 = \frac{a}{\pi}$, b' is an element of R_v satisfying $(\bar{b'})^{p^j} = \bar{b}$ and $b_1 = \frac{1}{\pi}(b + (-b')^{p^j})$;
(iii) when $a \notin M_v$, $b \in M_v$ and $v(n-m) = 0$, then $v(b) = v(\pi)$;
(iv) when $a \notin M_v$, $b \in M_v$ and $v(n-m) > 0$ with $l \geq 1$ as the highest power of p dividing $n - m$, then either $v(a_1) \geq v(\pi)$ and $v(b_2) = 0$ or $v(a_1) = 0 = v(b_2^{m-1}[(-a)^{m_1}(a_1)^{n_1-m_1} - (-b_2)^{n_1-m_1}])$, where $a_1 = \frac{1}{\pi}(a + (-a')^{p^l})$, $b_2 = \frac{b}{\pi}$, a' belonging to R_v satisfies $(\bar{a'})^{p^l} = \bar{a}$;
(v) when $ab \notin M_v$ and $m \in M_v$ with $n = s'p^k$, $m = sp^k$, p does not divide $gcd(s', s)$, then the polynomials $x^{s'} + ax^s + b$ and $\frac{1}{\pi}[ax^{sp^k} + b + (-a'x^s - b')^{p^k}]$ are coprime modulo M_v, where a', b' are in R_v satisfying $(\bar{a'})^{p^k} = \bar{a}$, $(\bar{b'})^{p^k} = \bar{b}$;

[1] We deal with only trinomials in this paper because they are a fairly tractable class of polynomials having a simple formula for discriminant.
[2] If $v(D) = 0$, then $\overline{F}(x)$ has no repeated factor and hence $R_v[\theta]$ is integrally closed by Theorem 1.A.

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