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Classifying bicrossed products of two Taft algebras

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ABSTRACT

We classify all Hopf algebras which factor through two Taft algebras $\mathbb{T}_{n^2}(\overline{q})$ and respectively $T_{m^2}(q)$. To start with, all possible matched pairs between the two Taft algebras are described: if $\overline{q} \neq q^{n-1}$ then the matched pairs are in bijection with the group of d-th roots of unity in k, where $d = (m, n)$ while if $\overline{q} = q^{n-1}$ then besides the matched pairs above we obtain an additional family of matched pairs indexed by k^*. The corresponding bicrossed products (double cross product in Majid's terminology) are explicitly described by generators and relations and classified. As a consequence of our approach, we are able to compute the number of isomorphism types of these bicrossed products as well as to describe their automorphism groups.

0. Introduction

The factorization problem originates in group theory and it was first considered by Maillet ([13]). Since then, it was also introduced and intensively studied in the context of other mathematical objects such as: (co)algebras ([6,7]), Lie algebras ([17]), Lie groups ([14]), groupoids ([4]), Hopf algebras ([16]), fusion categories ([9]) and so on. For a detailed historical update on the problem we refer to [2,3] and the references therein. The factorization problem in its original group setting asks for the description and classification of all groups X which factor through two given groups G and H, i.e. $X = GH$ and $G \cap H = \{1\}$. However, although the statement of the problem is very simple and natural, no major progress has been made so far as we still lack exhaustive methods to tackle it. For instance, even the description and classification of groups which factor through two finite cyclic groups is still an open problem although there are several papers dealing with it, such as the four papers by J. Douglas [8] and the more recent one [1] which solves completely the problem in the special case where one of the groups is of prime order.

One turning point in studying the factorization problem for groups was the bicrossed product construction introduced in a paper by Zappa ([20]); later on, the same construction appears in a paper of Takeuchi [18]

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where the terminology bicrossed product originates. The main ingredients in constructing bicrossed products are the so-called matched pairs of groups. The corresponding notions in the context of Hopf algebras (see Section 1 for the precise definitions) were introduced by Majid ([15]) and allowed for a more computational approach of the problem. This point of view was also considered recently in [2] where a strategy for classifying bicrossed products of Hopf algebras was proposed. The line of inquiry presented in [2] was followed in [5] where the Hopf algebras which factor through two Sweedler's Hopf algebras are described and classified as well as in [11] where the automorphism group of the Drinfel'd double of a purely non-abelian finite group is completely described. Similar ideas are also employed in [12] in order to determine all quasitriangular structures and ribbon elements on the Drinfel'd double of a finite group over an arbitrary field.

In this paper, using the method presented in [2], we investigate Hopf algebras which factor through two Taft algebras $T_{m^2}(q)$ and respectively $\mathbb{T}_{n^2}(\overline{q})$. More precisely, we will describe and classify all bicrossed products $T_{m^2}(q) \bowtie \mathbb{T}_{n^2}(\overline{q})$. The number of isomorphism types of these bicrossed products is also computed and, as expected, it depends heavily on the arithmetics of the base field k. In particular, for $m = n$ and $\overline{q} = q^{n-1}$, we find the celebrated Drinfel'd double $D\big(T_{n^2}(q)\big)$ which however is just one of the bicrossed products between two Taft algebras having the same dimension. As a consequence of our strategy the automorphism groups of all bicrossed products $T_{m^2}(q) \bowtie \mathbb{T}_{n^2}(\overline{q})$ are explicitly described. We mention as well that the problem of describing the automorphism group of a given Hopf algebra is a notoriously difficult problem coming from invariant theory.

The paper is organized as follows. In Preliminaries we review the construction of the bicrossed product associated to a matched pair of Hopf algebras $(A, H, \triangleleft, \triangleright)$. Then, we state Majid's result (Theorem 1.1) which turns the factorization problem for Hopf algebras into a computational one: given A and H two Hopf algebras, describe the set of all matched pairs $(A, H, \triangleright, \triangleleft)$ and classify up to an isomorphism all bicrossed products $A \bowtie H$. Section 2 gathers our main results. We start by describing all matched pairs $(\mathbb{T}_{n^2}(\overline{q}),\, T_{m^2}(q),\, \triangleright,\, \triangleleft)$ in Theorem 2.1. It turns out that if $\overline{q} \neq q^{n-1}$ then the matched pairs are in bijection with the group of d-th roots of unity in k, where $d = (m,\, n)$; if $\overline{q} = q^{n-1}$, then besides the matched pairs above we obtain an additional family of matched pairs indexed by k^*. The bicrossed products corresponding to the two families of matched pairs from Theorem 2.1 which we denote by $\mathrm{T}^{\sigma}_{n,\,m}(\overline{q},\, q)$ and respectively $\mathrm{Q}^{\alpha}_{n}(q)$, for some $\sigma \in U_d(k)$ and $\alpha \in k^*$, are described by generators and relations in Corollary 2.2. Our main classification result is Theorem 2.3 which gives necessary and sufficient conditions for any two bicrossed products between $\mathbb{T}_{n^2}(\overline{q})$ and $T_{m^2}(q)$ to be isomorphic. Several cases need to be considered depending on whether $n \neq m$ or $m = n$ and respectively $\overline{q} \neq q$ or $\overline{q} = q$. In particular, it is proved that the Drinfel'd double $D\big(T_{n^2}(q)\big)$ is isomorphic to $\mathrm{Q}^{1}_{n}(q)$.

Theorem 2.3 has two very important consequences. The first one is Corollary 2.5 which records the number of isomorphism types of the Hopf algebras described in Corollary 2.2 while the second one, Theorem 2.6, describes explicitly their automorphism groups. For instance the automorphism group of the Drinfel'd double $D\big(T_{n^2}(q)\big)$ is proved to be isomorphic to k^* for $n \geqslant 3$ and to a semidirect product $k^* \rtimes \mathbb{Z}_2$ for $n = 2$ (see also [5]).

1. Preliminaries

Throughout this paper, k will be an arbitrary field of characteristic zero. Unless otherwise specified, all algebras, coalgebras, bialgebras, Hopf algebras, tensor products and homomorphisms are over k. For a coalgebra C, we use Sweedler's Σ-notation: $\Delta(c) = c_{(1)} \otimes c_{(2)}$, $(I \otimes \Delta)\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, etc (summation understood). If C is a coalgebra, the opposite coalgebra, C^{cop}, has the same underlaying vector space C but with the comultiplication given by $\Delta^{cop} = \tau \circ \Delta$, where τ is the flip map $\tau(a \otimes b) = b \otimes a$. Given any vector space V we denote by V^* its dual space; if $v \in V$, then the corresponding element of V^* will be denoted simply by v^*. The notation $\delta_{i,\,j}$ stands for the Kronecker delta.

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