Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Right saturations and induced pseudofunctors between bicategories of fractions

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ARTICLE INFO

Article history:
Received 22 April 2015
Received in revised form 6 January 2017
Available online xxxx
Communicated by R. Vakil

MSC:
18A05; 18A30; 22A22

ABSTRACT

We fix any bicategory $\mathscr{A}$ together with a class of morphisms $\mathbf{W}_{\mathscr{A}}$, such that there is a bicategory of fractions $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ (as described by D. Pronk). Given another such pair $(\mathscr{B}, \mathbf{W}_{\mathscr{B}})$ and any pseudofunctor $\mathcal{F}: \mathscr{A} \to \mathscr{B}$, we find necessary and sufficient conditions in order to have an induced pseudofunctor $\mathcal{G}: \mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right] \to \mathscr{B}\left[\mathbf{W}_{\mathscr{B}}^{-1}\right]$. Moreover, we give a simple description of $\mathcal{G}$ in the case when the class $\mathbf{W}_{\mathscr{B}}$ is "right saturated".

1. Introduction

In order to make this Introduction concise, we assume that the reader is familiar with the standard notion of bicategory (in particular, the concept of vertical composition in a bicategory, denoted by $\odot$ below), as well as the notions of pseudofunctors between bicategories and pseudonatural equivalences between pseudofunctors. If not, we refer to § 2.1 for a short introduction to all these concepts.

In 1996 Dorette Pronk introduced the notion of (*right*) *bicalculus of fractions* (see [10]), generalizing the concept of (right) calculus of fractions (described in 1967 by Pierre Gabriel and Michel Zisman, see [2]) from the framework of categories to that of bicategories. Pronk proved that given a bicategory $\mathscr{C}$ together with a class of morphisms $\mathbf{W}$ (satisfying a set of technical conditions called (BF)), there are a bicategory $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ (called (*right*) *bicategory of fractions*) and a pseudofunctor $\mathcal{U}_{\mathbf{W}}: \mathscr{C} \to \mathscr{C}\left[\mathbf{W}^{-1}\right]$. This pseudofunctor sends each element of $\mathbf{W}$ to an internal equivalence and is universal with respect to this property (see [10, Theorem 21]). The structure of $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ depends on a set of choices $\mathrm{C}(\mathbf{W})$ involving axioms (BF) (see § 2.2); by the universal property of $\mathcal{U}_{\mathbf{W}}$, different sets of choices give rise to equivalent bicategories.

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http://dx.doi.org/10.1016/j.jpaa.2017.03.014
0022-4049/

If you are not familiar with the construction of objects, morphisms and 2-morphisms in a bicategory of fractions, we refer to [10, § 2.2 and 2.3]. § 2.2 below provides an introduction to all the relevant notions and notation in this context, that will be used heavily in the statement of Theorem 1.3 below.

Now let us suppose that we have two pairs $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ and $(\mathscr{B}, \mathbf{W}_{\mathscr{B}})$, both admitting a right bicalculus of fractions, and any pseudofunctor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$. Then the following three questions arise naturally:

(a) what are the necessary and sufficient conditions such that there are a pseudofunctor $\mathcal{G}$ and a pseudonatural equivalence κ as in the following diagram?

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\quad \mathcal{F} \quad} & \mathscr{B} \\ {\scriptstyle \mathcal{U}_{\mathbf{W}_{\mathscr{A}}}} \big\downarrow & \swarrow\!\!\swarrow\ \kappa & \big\downarrow {\scriptstyle \mathcal{U}_{\mathbf{W}_{\mathscr{B}}}} \\ \mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right] & \dashrightarrow_{\mathcal{G}} & \mathscr{B}\left[\mathbf{W}_{\mathscr{B}}^{-1}\right] \end{array} \tag{1.1}$$

(b) If a pair $(\mathcal{G}, \kappa)$ as above exists, can we express $\mathcal{G}$ in a simple form, at least in some cases?
(c) Again if $(\mathcal{G}, \kappa)$ as above exists, what are the necessary and sufficient conditions on $\mathcal{F}$, such that $\mathcal{G}$ is an equivalence of bicategories?

The typical case where these three questions arise naturally is that of stacks. For example by [10] it is known that the 2-category of differentiable stacks can be presented as a bicategory of fractions, starting from the 2-category $\mathscr{B}$ of étale Lie groupoids and "localizing" the class $\mathbf{W}_{\mathscr{B}}$ of essential equivalences (analogous results hold for the 2-category of topological stacks and for the one of algebraic stacks). In various concrete applications, one does not want to work with (Lie) groupoids (nor directly with stacks), so if we want to give an alternative description of (differentiable, topological, algebraic) stacks, then a tentative general plan is the following. We should firstly describe a bicategory $\mathscr{A}$, suitable for the applications we have in mind, then find a class $\mathbf{W}_{\mathscr{A}}$ and a pseudofunctor $\mathcal{F}$ as above. Finally, we should solve questions (a)–(c) in this framework. We will be more precise on this kind of application in the last section of this paper. Instead of solving these problems only in the case of (Lie, topological or algebraic) groupoids, we want to find the solution for this problem in the generality stated above. An explicit application of these results is provided in our paper [17].

We are going to give an answer to (a) and (b) in this paper, while an answer to (c) will be given in the next paper [16]. In order to prove all these results, a key notion will be that of (*right*) *saturation*: given any pair $(\mathscr{C}, \mathbf{W})$ as above, we define the (right) saturation $\mathbf{W}_{\text{sat}}$ of $\mathbf{W}$ as the class of all morphisms $f : B \to A$ in $\mathscr{C}$, such that there are a pair of objects C, D and a pair of morphisms $g : C \to B$, $h : D \to C$, such that both $f \circ g$ and $g \circ h$ belong to $\mathbf{W}$. If $(\mathscr{C}, \mathbf{W})$ satisfies conditions (BF), then $\mathbf{W} \subseteq \mathbf{W}_{\text{sat}}$ and $\mathbf{W}_{\text{sat}} = \mathbf{W}_{\text{sat,sat}}$, thus explaining the name "saturation" for this class. Moreover, we have the following key result:

Proposition 1.1. (Lemma 3.10 and Proposition 3.12) *Let us fix any pair $(\mathscr{C}, \mathbf{W})$ satisfying conditions* (BF). *Then also the pair $(\mathscr{C}, \mathbf{W}_{\text{sat}})$ satisfies the same conditions, so there are a bicategory of fractions $\mathscr{C}\left[\mathbf{W}_{\text{sat}}^{-1}\right]$ and a pseudofunctor*

$$\mathcal{U}_{\mathbf{W}_{\text{sat}}} : \mathscr{C} \longrightarrow \mathscr{C}\left[\mathbf{W}_{\text{sat}}^{-1}\right] \tag{1.2}$$

with the universal property. Moreover, there is an equivalence of bicategories $\mathcal{H} : \mathscr{C}\left[\mathbf{W}_{\text{sat}}^{-1}\right] \to \mathscr{C}\left[\mathbf{W}^{-1}\right]$ and a pseudonatural equivalence of pseudofunctors $\tau : \mathcal{U}_{\mathbf{W}} \Rightarrow \mathcal{H} \circ \mathcal{U}_{\mathbf{W}_{\text{sat}}}$.

Then the answer to question (a) is given by the equivalence of (i) and (iii) below.