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On a conjecture about orders of products of elements in the symmetric group

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A R T I C L E I N F O

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A B S T R A C T

For any $a, b, c, n \in \mathbb{N}$ with $1 < a, b, c \leq n-2$, then there exist $\alpha, \beta \in S_n$ such that $o(\alpha) = a$, $o(\beta) = b$ and $o(\alpha\beta) = c$. This is a conjecture of Stefan Kohl and which is closely related to problem on covers of the complex projective line. In this note we prove the conjecture is true.

1. Introduction

In [3, problem 18.49], Stefan Kohl proposed a conjecture that for any $a, b, c \in \mathbb{N}$ with $1 < a, b, c \leq n-2$, then, there exist $\alpha, \beta \in S_n$ such that $o(\alpha) = a$, $o(\beta) = b$ and $o(\alpha\beta) = c$. J. König proved the conjecture is true in [2], however, our proofs are very different. In fact, we solved this conjecture independently.

Let S_n be the symmetric group on the set $[n] = \{1, 2, \cdots, n\}$ and $\alpha \in S_n$. Then there exist disjoint cycles $\alpha_1, \alpha_2, \cdots, \alpha_k$ in S_n, where α_i be r_i-cycle for $i = 1, 2, \cdots, k$ such that $\alpha = \alpha_1\alpha_2 \cdots \alpha_k$, and furthermore the order of α is the least common multiple of $r_1, r_2, \cdots, r_k$, denoted by $o(\alpha) = [r_1, r_2, \cdots, r_k]$. Hence, the conjecture is closely related to product of cycles. Some properties on cycles have been found, for detail see [1].

In this note, we firstly prove some properties on the products of cycles and then apply them to construct α and β in S_n such that $o(\alpha) = a$, $o(\beta) = b$ and $o(\alpha\beta) = c$ for any $a, b, c, n \in \mathbb{N}$ with $1 < a, b, c \leq n-2$, and thus we obtain following theorem.

Theorem 1.1. *Let $n, a, b, c \in \mathbb{N}$ with $1 < a, b, c \leq n-2$. Then, there exist α and β in S_n such that $o(\alpha) = a$, $o(\beta) = b$ and $o(\alpha\beta) = c$.*

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2. Preliminary lemmas

Throughout this paper, if there is no special statement, the different letters indicate different points and the same letter with different marks also indicate different points. Furthermore, (1) denotes the identity of S_n. To express theorems clearly, we firstly introduce following definitions.

Definition 2.1. Let $\alpha = (x, y, \cdots, z)$ be a r-cycle in S_n. If we set $x_1 = x, x_2 = y, \cdots, x_r = z$, then $\alpha = (x_1, x_2, \cdots, x_r)$ and we say this is a cycle label of α.

It is clear that if we set $y_1 = y, \cdots, y_{r-1} = z, y_r = x$, then, $\alpha = (y_1, y_2, \cdots, y_r)$ is also a cycle label of α, and furthermore there are r kinds of cycle labels of a r-cycle.

Definition 2.2. Let α be permutation in S_n. Then, $\{\alpha\} = \{x \in [n] | x^\alpha \neq x\}$.

An easy computation to show that the following theorems hold. Although the proofs are trivial, these results play an important role in proving the Theorem 1.1.

Theorem 2.3. *Let α and β be two cycles in S_n with $\{\alpha\} \cap \{\beta\} = \{z_1, z_2, \cdots, z_t\}$ and t is an odd number. If there exist cycle labels $\alpha = (x_1, x_2, \cdots, x_k)$, $\beta = (y_1, y_2, \cdots, y_l)$ and $x_{i_1} = y_{j_1} = z_1, \cdots, x_{i_t} = y_{j_t} = z_t$ such that $1 \leq i_1 < i_2 < \cdots < i_t \leq k$, $1 \leq j_1 < j_2 < \cdots < j_t \leq l$, then $\alpha\beta$ is also a cycle and the length of $\alpha\beta$ is $k + l - t$.*

Theorem 2.4. *Let α and β be two cycles in S_n and $|\{\alpha\} \cap \{\beta\}| = s + t$ and t is an even number. If there exist cycle labels $\alpha = (x_1, x_2, \cdots, x_k)$ and $\beta = (y_1, y_2, \cdots, y_l)$ such that $y_1 = x_s, y_2 = x_{s-1}, \cdots, y_s = x_1, y_{j_1} = x_{i_1}, \cdots, y_{j_t} = x_{i_t}$, $i_1 < i_2 < \cdots < i_t$, $j_1 < j_2 < \cdots < j_t$, then $\alpha\beta$ is a $(k + l - 2s - t + 1)$-cycle and $x_1, \cdots, x_{s-1}$ are fixed points of $\alpha\beta$.*

Theorem 2.5. *Let α and β be two cycles in S_n and $|\{\alpha\} \cap \{\beta\}| = t + 2$ and t is an even number. If there exist cycle labels $\alpha = (x_1, x_2, \cdots, x_k)$ and $\beta = (y_1, y_2, \cdots, y_l)$ such that $y_1 = x_s, y_2 = x_1, y_{j_1} = x_{i_1}, \cdots, y_{j_t} = x_{i_t}$, $s < i_1 < i_2 < \cdots < i_t$, $j_1 < j_2 < \cdots < j_t$, then $\alpha\beta$ is the product of two disjoint cycles, one is $(x_1, x_2, \cdots, x_{s-1})$, the other is a $(k + l - s - t - 1)$-cycle. In particular, if $s = 3$, one is a 2-cycle.*

3. Proof of Theorem 1.1

In this section we apply above theorems to construct α and β with $o(\alpha) = a$ and $o(\beta) = b$ in S_n such that $\alpha\beta$ is either a c-cycle or the product of a c-cycle and a 2-cycle.

Suppose that the Theorem 1.1 is true for $1 < b \leq a \leq c \leq n - 2$. Then there exist β' and γ in S_n such that $o(\beta') = b$, $o(\gamma) = c$ and $o(\beta'\gamma) = a$ for $1 < b \leq c \leq a \leq n - 2$. Pick $\alpha = \beta'\gamma$ and $\beta = \beta'^{-1}$. It is easy to show that $o(\alpha) = a$, $o(\beta) = b$ and $o(\alpha\beta) = c$, and so the Theorem 1.1 still holds for this case. Similarly, we prove that the Theorem 1.1 is also true for other cases. Hence, we may assume that $1 < b \leq a \leq c$ and $c = ka + r$, $1 \leq k$, $0 \leq r < a$.

In what follows, we distinguish three cases to construct α and β, respectively as $r = 0$, b is an even number with $r > 0$ and b is an odd number with $r > 0$. And we shall construct α and β with $|\{\alpha\} \bigcup \{\beta\}| \leq c + 2$ for each case, which implies that α and β are both in S_n. For each case, $o(\alpha) = a$, $o(\beta) = b$ and $|\{\alpha\} \bigcup \{\beta\}| \leq c+2$ are clear and so we do not note one by one.

Lemma 3.1. *Let $n, a, b, c \in \mathbb{N}$ with $1 < b \leq a \leq c \leq n - 2$ and $c = ka$, $k \geq 1$. Then there exist α and β in S_n such that $o(\alpha) = a$, $\beta = b$ and $o(\alpha\beta) = c$.*