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The additive completion of the biset category

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ABSTRACT

Let R be a commutative unital ring. We construct a category $\mathscr{C}_R$ of *fractions* X/G, where G is a finite group and X is a finite G-set, and with morphisms given by R-linear combinations of spans of bisets. This category is an additive, symmetric monoidal and self-dual category, with a Krull–Schmidt decomposition for objects. We show that $\mathscr{C}_R$ is equivalent to the additive completion of the biset category and that the category of biset functors over R is equivalent to the category of R-linear functors from $\mathscr{C}_R$ to R-Mod. We also show that the restriction of one of these functors to a certain subcategory of $\mathscr{C}_R$ is a fused Mackey functor.

1. Introduction

The aim of this paper is to give an explicit description of the additive completion of the biset category and to show some of its properties.

The additive completion of the biset category will be equivalent to the category $\mathscr{C}_R$, the objects of which are couples (G, X) where G is a finite group and X is a finite (left) G-set. These objects will be written as fractions $\frac{X}{G}$ for convenience (or as X/G in case of limited space). One of the nice features of this notation is that $\mathscr{C}_R$ is an additive symmetric monoidal category with addition and tensor product given by

$$\frac{X}{G} \oplus \frac{Y}{H} = \frac{(X \times H) \sqcup (G \times Y)}{G \times H} \quad \text{and} \quad \frac{X}{G} \otimes \frac{Y}{H} = \frac{X \times Y}{G \times H}.$$

The arrows and the composition in $\mathscr{C}_R$ will be explained in detail in the next section.

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We will see that objects of the form $\{\bullet\}/G$ are indecomposable in $\mathscr{C}_R$ and that every object in $\mathscr{C}_R$ can be written as a sum of this kind of objects in a unique way, up to isomorphism. This gives a Krull–Schmidt decomposition for objects in $\mathscr{C}_R$, although this does not make $\mathscr{C}_R$ a Krull–Schmidt category, in the sense of the formal definition, because the endomorphism ring of $\{\bullet\}/G$ in $\mathscr{C}_R$ is isomorphic to the double Burnside ring $RB(G, G)$, which in general is not a local ring. We will also show that $\mathscr{C}_R$ is a self-dual category and that it has a Mackey decomposition for arrows.

In the last section we consider R-linear functors from $\mathscr{C}_R$ to R-Mod and their relation with Mackey and biset functors. First we will prove that for a finite group G we have a functor from the Burnside category (the category having as objects the isomorphism classes of finite G-sets and as arrows the spans of G-sets) to $\mathscr{C}_R$. This functor is injective in objects but it is not full, nor faithful. Nevertheless, we will see that by pre-composition with this functor, an R-linear functor from $\mathscr{C}_R$ to R-Mod gives a *fused Mackey functor*, as defined in [3]. On the other hand, we prove that the category of biset functors is equivalent to the category of R-linear functors from $\mathscr{C}_R$ to R-Mod. As an example, we see that the Burnside biset functor extends to the functor from $\mathscr{C}_R$ to R-Mod which sends an object X/G to the Burnside group $B(X)$.

This paper originates in results appearing in the PhD thesis of the first author [5]. A related, more general, construction to our category $\mathscr{C}_R$ is given by Nakaoka [8].

2. Preliminaries

In this section G, H and K will be finite groups.

2.1. About G-sets

We will write $B(G)$ for the Burnside group (ring) of finite G-sets.

Recall, for example from Section 1.6 in Bouc [1], that given a G-set X, a couple (Y, φ) is called *a G-set over X* if Y is a G-set and φ is a morphism of G-sets from Y to X. A morphism from (Y, φ) to (Y', φ') is a morphism of G-sets $f : Y \to Y'$ such that $\varphi' f = \varphi$. Whit this data one can define the category of G-sets over X, denoted by $G\text{-set}{\downarrow_X}$. The Grothendieck group of the category $G\text{-set}{\downarrow_X}$, for relations given by decomposition into disjoint union, is denoted by $B(X)$. It is easy to see that $B(X)$ has an additive structure and it is thus called the *Burnside group of the G-set X*. By Proposition 2.4.2 in [1], if X is a transitive G-set, isomorphic to a set of left cosets G/H, then $B(X)$ coincides with $B(H)$, the Burnside group of the group H, this will also be a consequence of Lemma 4.2. This construction can be endowed with a structure of Green functor, for more details see Section 2.4 in [1]. In particular, $B(X)$ has a multiplicative structure, given explicitly by the pullback in the category G-set, extended by bilinearity.

The *Burnside category* (see for example Lindner [6]), denoted by *Span*(G-set) or $Span_R$(G-set) if we are taking coefficients in a commutative unital ring R, is the category of spans in the category G-set, namely: objects in *Span*(G-set) are finite G-sets, and the set of morphisms from a G-set X to a G-set Y, denoted by $B^G(Y \times X)$, is defined as follows. The set $Y \times X$ has a natural structure of G-set, so we consider the elements of the form $(S, \beta \times \alpha)$ in $B(Y \times X)$, where

$$\begin{tikzcd} & S \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ Y & & X \end{tikzcd}$$

is a span in G-set and $(\beta \times \alpha)(s) = (\beta(s), \alpha(s))$. Then $B^G(Y \times X)$ is generated by the equivalence classes of these elements under the relation which makes $Y \leftarrow S \to X$ equivalent to $Y \leftarrow U \to X$ if there is an isomorphism of G-sets between S and U which commutes with the projection maps to X and Y. Composition

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