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A class of retracts of polynomial algebras

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A B S T R A C T

We show that all retractions of polynomial algebras with sparse homogeneous parts are conjugate to the canonical ones, and as a consequence, the corresponding retracts are isomorphic to polynomial algebras.

1. Introduction

Throughout the paper, k denotes a field of characteristic zero unless otherwise mentioned, and $k^{[n]} := k[x_1, \dots, x_n]$ stands for the polynomial algebra in n variables over k. A subalgebra R of a k-algebra S is called a retract if it is the image of some retraction (i.e., idempotent endomorphism) of S. In the category of commutative k-algebras, a k-algebra P is a projective object if and only if P is a retract of some polynomial algebra in not necessarily finite number of variables.

The study of retracts of polynomial algebras is closely related to some problems in affine algebraic geometry. For example, Shpilrain and Yu [7] proved that the 2-dimensional Jacobian conjecture is equivalent to the following conjecture.

Conjecture. *If for a pair of polynomials $p, q \in k[x_1, x_2]$ the corresponding Jacobian matrix is invertible, then $k[p]$ is a retract of $k[x_1, x_2]$.*

Retracts were applied to the automorphic orbit problem for polynomial algebras in two variables, see [2,11]. And by use of retracts, the second author [8] gave a new method to describe automorphisms of the endomorphism semigroups of free algebras including polynomial algebras and free Poisson algebras. Retracts were also involved with Zariski's cancellation problem.

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Zariski's cancellation problem (ZCP for short; see [6, Chapter 6]) *Let A be a commutative k-algebra satisfying $A^{[1]} \cong k^{[n]}$. Then does it follow that $A \cong k^{[n-1]}$?*

The ZCP has an affirmative answer for $n \leq 3$ and it is still open for all $n \geq 4$, see [9] for a survey. (Note that Gupta [3,4] showed that if char $k > 0$ then the ZCP has a negative answer for all $n \geq 4$.)

In this paper, we focus on the following problem concerning retracts.

Problem 1.1. Is every proper retract of the polynomial algebra $k^{[n]}$ isomorphic to a polynomial algebra?

An affirmative answer to Problem 1.1 would give a positive solution to the ZCP: the isomorphism $A^{[1]} \cong k^{[n]}$ implies A is isomorphic to some retract of $k^{[n]}$; if Problem 1.1 has an affirmative answer, then $A \cong k^{[r]}$ for some r, and since $r + 1 = \dim A^{[1]} = \dim k^{[n]} = n$, one finally gets the isomorphism $A \cong k^{[n-1]}$.

Costa [1] showed that every proper retract of $k^{[2]}$ is isomorphic to $k^{[1]}$, i.e., is of the form $k[p]$ for some non-constant polynomial $p \in k^{[2]}$. Furthermore, Shpilrain and Yu [7] showed that there exists an automorphism ϕ of $k^{[2]}$, such that $\phi(p) = x_1 + x_2 q$ for some $q \in k^{[2]}$. Problem 1.1 is still open for all $n \geq 3$, and up to now few results are known on the structure of retracts of $k^{[n]}$ for $n \geq 3$.

In this paper, we introduce the concept of a retraction with sparse homogeneous parts (see Definition 2.1 below) and show that such a retraction is conjugate to the canonical retraction $(x_1+h_1, \ldots, x_r+h_r, 0, \ldots, 0)$, where each h_i belongs to the ideal of $k^{[n]}$ generated by $x_{r+1}, \ldots, x_n$. This implies that the corresponding retract is isomorphic to a polynomial algebra (Theorem 2.2). As a corollary, a retraction of $k^{[n]}$ of the form

$$\phi = \phi^{(1)} + \phi^{(a)} + \cdots + \phi^{(2a-2)}, \quad a \geq 2, \tag{1}$$

where $\phi^{(i)}$ denotes the homogeneous part of degree i of ϕ, is conjugate to a canonical retraction (Corollary 2.3). This result can be seen as an analogue of a result of Jurkiewicz in [5], which says that an automorphism of order 2 of $k^{[n]}$ of the form (1) is conjugate to a linear automorphism.

2. Retractions with sparse homogeneous parts and the corresponding retracts

We start with some notations and basic facts concerning endomorphisms of polynomial algebras. A k-algebra endomorphism ϕ of $k^{[n]} = k[x_1, \ldots, x_n]$ will be written as $\phi = (\phi_1, \ldots, \phi_n)$, where $\phi_i = \phi(x_i)$. For two endomorphisms $\phi = (\phi_1, \ldots, \phi_n)$ and $\psi = (\psi_1, \ldots, \psi_n)$, the composition is $\phi \circ \psi = (\psi_1(\phi_1, \ldots, \phi_n), \ldots, \psi_n(\phi_1, \ldots, \phi_n))$, and the sum is defined to be $\phi + \psi := (\phi_1 + \psi_1, \ldots, \phi_n + \psi_n)$, i.e., the endomorphism which maps x_i to $\phi_i + \psi_i$. Note that $\mu \circ (\phi + \psi) = \mu \circ \phi + \mu \circ \psi$ for any endomorphism μ, and $(\phi + \psi) \circ \mu = \phi \circ \mu + \psi \circ \mu$ for any linear endomorphism μ.

For a polynomial f, we denote by $f^{(d)}$ the homogeneous part of degree d of f. And for an endomorphism ϕ, we denote by $\phi^{(d)} = (\phi_1^{(d)}, \ldots, \phi_n^{(d)})$ the homogeneous part of degree d of ϕ, and then $\phi = \sum_d \phi^{(d)}$.

Let ϕ be a retraction of $k^{[n]}$. Observe that for any automorphism ψ of $k^{[n]}$, the conjugation $\widetilde{\phi} := \psi \circ \phi \circ \psi^{-1}$ is also a retraction of $k^{[n]}$, and $\widetilde{\phi}(k^{[n]}) = \psi(\phi(k^{[n]})) \cong \phi(k^{[n]})$. Since ϕ is a retraction, we have $\phi_i(\phi_1, \ldots, \phi_n) = \phi_i$, and thus $\phi_i(\phi_1(0), \ldots, \phi_n(0)) = \phi_i(0)$. If we let $\rho = (x_1 + \phi_1(0), \ldots, x_n + \phi_n(0))$, then $(\rho \circ \phi \circ \rho^{-1})(x_i) = \phi_i(x_1 + \phi_1(0), \ldots, x_n + \phi_n(0)) - \phi_i(0)$ whose constant term is $\phi_i(\phi_1(0), \ldots, \phi_n(0)) - \phi_i(0) = 0$. In conclusion, up to conjugation, one may assume that $\phi^{(0)} = 0$.

Definition 2.1. Let ϕ be an endomorphism of $k^{[n]}$ with $\phi^{(0)} = 0$. We say that ϕ has sparse homogeneous parts if there exist positive integers a_i, b_i, $i = 1, \ldots, t$ such that

$$\phi = \phi^{(1)} + \sum_{d_1 \in [a_1, b_1]} \phi^{(d_1)} + \sum_{d_2 \in [a_2, b_2]} \phi^{(d_2)} + \cdots + \sum_{d_t \in [a_t, b_t]} \phi^{(d_t)},$$

where