Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Green's theorem and Gorenstein sequences

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A R T I C L E   I N F O

Article history:
Received 8 October 2016
Received in revised form 23 March 2017
Available online xxxx
Communicated by S. Iyengar

MSC:
Primary 13D40; secondary 13H10; 14C20

A B S T R A C T

We study consequences, for a standard graded algebra, of extremal behavior in Green's Hyperplane Restriction Theorem. First, we extend his Theorem 4 from the case of a plane curve to the case of a hypersurface in a linear space. Second, assuming a certain Lefschetz condition, we give a connection to extremal behavior in Macaulay's theorem. We apply these results to show that $(1, 19, 17, 19, 1)$ is not a Gorenstein sequence, and as a result we classify the sequences of the form $(1, a, a-2, a, 1)$ that are Gorenstein sequences.

1. Introduction

In the study of Hilbert functions of standard graded algebras, Macaulay's theorem [18] and Green's theorem [15] stand out as being of fundamental importance both on a theoretical level and from the point of view of applications. Macaulay's theorem regulates the possible growth of the Hilbert function from one degree to the next. It is a stunning fact that strong geometric consequences arise whenever the maximum possible growth allowed by this theorem is achieved [14,7,2], or even when the maximum is *almost* achieved [10]. Green's theorem regulates the possible Hilbert functions of the restriction modulo a general linear form. It is a less-studied question to ask what happens if the maximum possible Hilbert function occurs for this restriction, although already Green gave some intriguing results [15,8] in his so-called "Theorem 3" and "Theorem 4," and some results in this direction can also be found in [1]. To our knowledge, the connections between these two kinds of extremal behavior have not previously been studied.

One area where both Macaulay's theorem and Green's theorem have been applied very profitably is the problem of classifying the Hilbert functions of Artinian Gorenstein algebras (i.e. of finding all possible *Gorenstein sequences*). Of course this problem is probably intractable in full generality. However, many papers have been written on the subject, and we cannot begin to list them all here. Even the special case

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http://dx.doi.org/10.1016/j.jpaa.2017.04.010

of socle degree 4 (i.e. Gorenstein sequences of the form $(1, a, b, a, 1)$) has been carefully studied (see for instance [27,24,3,8,25,4]), but a full classification remains open.

Given the ubiquity of Gorenstein rings [5], the problem of classifying the possible Gorenstein sequences is of intrinsic interest. However, the unimodality question also has a strong motivation coming from geometry and combinatorics. As noted in [23], the Stanley–Reisner ring of the boundary complex of a convex polytope is a reduced Gorenstein ring, and the g-theorem classifies their h-vectors as being the so-called SI-sequences. These sequences are defined as being unimodal symmetric sequences such that the first difference of the "first half" is an O-sequence. It was shown in [22] that over a field of characteristic zero, every SI-sequence is even the h-vector of a simplicial polytope with whose Stanley–Reisner ring has maximal Betti numbers among simplicial polytopes with the given h-vector. The SI-property is proven (following Stanley) by proving that the weak Lefschetz property (WLP) holds for that ring. It is an open question whether the same result holds for simplicial spheres; this would follow from an affirmative answer to Question 1.4 of [22], namely whether every reduced, arithmetically Gorenstein subscheme of projective space possesses the WLP. A negative answer would arise by producing an Artinian Gorenstein algebra that "lifts" to a reduced set of points but does not have a unimodal h-vector. Hence we are motivated to study Artinian Gorenstein algebras whose Hilbert function is not unimodal.

In this paper we make progress on both problems. First, we study some consequences of extremality for Green's theorem, including an analysis of a situation where we have an equivalence between this extremality and that for Macaulay's theorem. Next we apply this work to produce new results on Gorenstein sequences of socle degree 4.

More precisely, after recalling known facts in section 2, our main goal in section 3 is to find new consequences of extremal behavior in Green's theorem. We recall Green's Theorem 4 and we first prove a direct generalization in Theorem 3.2, passing from Green's case of a plane curve to the case of a hypersurface in a linear subspace. Our main result in this section is Theorem 3.5, which gives a connection, under certain assumptions, between extremal behavior for Green's theorem and extremal behavior for Macaulay's theorem. Because of this connection, Gotzmann's theorem applies as it did in the paper [7] to give strong geometric consequences, which we explore in Corollary 3.7. We also show that Green's theorem is "sequentially sharp" in Corollary 3.11.

An important feature of our work is a study of the geometric and algebraic consequences resulting from certain assumptions on the end of the binomial expansion of some term of the Hilbert function, if Green's theorem is sharp in that degree. See for instance the conditions $e \geq 2$ of Theorem 3.5 or $a_e > e$ of Corollary 3.7 (iv). It was pointed out to us by the referee that this feature could be a motivation for further studies on this topic.

We apply our new results on Green's theorem in Section 4 to show that the sequence $(1, 19, 17, 19, 1)$ is not Gorenstein (Theorem 4.1). Our proof brings together a number of different techniques. The result is the main ingredient for our Corollary 4.3, which completes the classification of the socle degree 4 Gorenstein sequences with $a - b = 2$ (with the notation introduced above) by proving that the sequence is Gorenstein if and only if $a \geq 20$.

Theorem 3.5 makes a certain numerical assumption as well as a certain Lefschetz assumption in order to conclude that the two different kinds of extremal behavior are equivalent. This gives a new illustration of the importance of the so-called Lefschetz properties, which have been studied very extensively in the last two decades, especially the Weak Lefschetz Property (WLP) and the Strong Lefschetz property (SLP). However, it is worth noting here that our Lefschetz assumption is much milder than WLP. Instead, we only assume that multiplication on our algebra by a general linear form is injective in just one degree. Interestingly, there are two different degrees where such an assumption leads to the equivalence mentioned above. This Lefschetz (injectivity) assumption can be phrased in more than one way, as shown in Lemma 3.4. It also leads to a surprisingly simple but useful result, Lemma 3.12, which forces the existence of a socle element in a specific degree. It is a small improvement of [20, Proposition 2.1 (b)], although our proof is completely

Please cite this article in press as: J. Ahn et al., Green's theorem and Gorenstein sequences, J. Pure Appl. Algebra (2017), http://dx.doi.org/10.1016/j.jpaa.2017.04.010