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Curves and envelopes that bound the spectrum of a matrix

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A B S T R A C T

A generalization of the method developed by Adam, Psarrakos and Tsatsomeros to find inequalities for the eigenvalues of a complex matrix A using knowledge of the largest eigenvalues of its Hermitian part $H(A)$ is presented. The numerical range or field of values of A can be constructed as the intersection of half-planes determined by the largest eigenvalue of $H(e^{i\theta}A)$. Adam, Psarrakos and Tsatsomeros showed that using the two largest eigenvalues of $H(A)$, the eigenvalues of A satisfy a cubic inequality and the envelope of such cubic curves defines a region in the complex plane smaller than the numerical range but still containing the spectrum of A. Here it is shown how using the three largest eigenvalues of $H(A)$ or more, one obtains new inequalities for the eigenvalues of A and new envelope-type regions containing the spectrum of A.

1. Introduction

In this paper, we denote by $\mathcal{M}_{n,k}(\mathbb{C})$ and $\mathcal{M}_{n,k}(\mathbb{R})$ the spaces of complex and real $n \times k$ matrices respectively; $\mathcal{M}_n(\mathbb{C})$ and $\mathcal{M}_n(\mathbb{R})$ stand for $k = n$. The spectrum $\sigma(A)$ of a matrix $A \in \mathcal{M}_n(\mathbb{C})$ is known to be located in its numerical range or field of values

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$F(A) = \{\mathbf{x}^* A\mathbf{x} \in \mathbb{C};\ \mathbf{x} \in \mathbb{C}^n, ||\mathbf{x}||_2 = 1\}$. The spectrum of A is also located to the left of the vertical line $Re(z) = \delta_1$ in the complex plane, where δ_1 is the largest eigenvalue of the Hermitian part $H(A) = \frac{1}{2}(A + A^*)$ of A. Here A^* denotes the Hermitian conjugate of A. Also, $S(A) = \frac{1}{2}(A - A^*)$ denotes the skew-Hermitian part of A, so $A = H(A) + S(A)$.

Clearly $F(A) = e^{-i\theta}F(e^{i\theta}A)$ for any $\theta \in [0, 2\pi[$, so $e^{i\theta}F(A)$ is located to the left of the vertical line $Re(z) = \lambda_{max}(H(e^{i\theta}A))$, and rotating this line by $e^{-i\theta}$ we get a new line that bounds $\sigma(A)$. In fact, [2,3], $F(A)$ is obtained exactly as the connected, compact and convex region defined by the envelope of all such lines. In the first subfigure of Fig. 1 we have illustrated this for the Toeplitz matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & i \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 \end{pmatrix} , \tag{1}$$

by plotting these lines for $\theta = 2\pi m/120, m = 0, \ldots, 119$. The eigenvalues of A are marked by small boxes in the figure.

In [1] Adam and Tsatsomeros showed how one can use the two largest eigenvalues δ_1 and δ_2 of $H(A)$ and the eigenvector $\mathbf{u}_1$ of $H(A)$ corresponding to δ_1, to obtain an improved inequality for $\sigma(A)$. Let $\alpha = Im(\mathbf{u}_1^* S(A)\mathbf{u}_1)$ and $K_1 = ||S(A)\mathbf{u}_1||_2^2 - \alpha^2 \geq 0$. Then they proved that any $\lambda \in \sigma(A)$ satisfies

$$|\lambda - (\delta_1 + i\alpha)|^2(Re(\lambda) - \delta_2) \leq K_1(\delta_1 - Re(\lambda)) . \tag{2}$$

With equality in (2) we have a curve $\Gamma_1(A)$ that bounds $\sigma(A)$ and is of degree 3 in the real and imaginary parts of λ. Applied to $H(e^{i\theta}A)$ one can repeat the argument above and obtain rotated cubic curves that bound $\sigma(A)$. The envelope of such curves was studied extensively by Psarrakos and Tsatsomeros [5,6] and they showed that it bounds a region $\mathcal{E}_1(A)$ that contains $\sigma(A)$ and is compact, but which is not always convex or connected. In the second subfigure of Fig. 1 we show their image [5] of $\mathcal{E}_1(A)$ for the Toeplitz matrix A in (1). Again we used 120 curves, with $\theta = 2\pi m/120, m = 0, \ldots, 119$, for the plot.

The aim of this paper is to generalize the results of Adam, Psarrakos and Tsatsomeros by using the k largest eigenvalues of $H(A)$ to obtain new curves $\Gamma_k(A)$ that bound $\sigma(A)$. As a preview of our results, we show in the third subfigure of Fig. 1 the region $\mathcal{E}_2(A)$ obtained from the envelope of curves when the three largest eigenvalues of $H(A)$ are utilized for the Toeplitz matrix A in (1). Also here 120 curves are used to construct the figure.

In Section 2 we derive the main inequality for the eigenvalues of a matrix, with respect to the largest eigenvalues and corresponding eigenvectors of its Hermitian part. We also state a more explicit formulation for the case of three known eigenvalues, and present some illustrations and analyze properties of the curves that bound the spectrum. In Section 3 we analyze explicitly some cases where the curves have special properties. Then, in Section 4, we demonstrate how an envelope of such curves encloses a region