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Projections on von Neumann algebras as limits of elementary operators

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ABSTRACT

If B is a subalgebra of a von Neumann algebra $A \subset \mathcal{B}(H)$ and B contains the rank one projections corresponding to an orthonormal basis of H, then a linear B-bimodule projection P on A with range B is of the form

$$P(x) = \sum_j p_j x p_j \quad x \in \mathcal{B}(H)$$

for orthogonal projections p_j in A which are diagonal with respect to the basis. An analogous result holds if $A = \mathcal{B}(H)$ and B is a weakly closed ternary ring of operators.

1. Introduction

In ring theory and in the context of algebras, idempotents have many well-established uses. In particular, if $e \in R$ is an idempotent of a ring R, then the subring eRe has unit e and there is an eRe-bimodule projection $x \mapsto exe$ from R onto eRe. The kernel

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$eR(1-e)+(1-e)Re+(1-e)R(1-e)$ of the projection is a complementary eRe-submodule of R.

In probability theory, and in the theory of von Neumann algebras, the notion of conditional expectation (as a completely positive map $E : M \to M$ on a von Neumann algebra M, with M commutative in the case of probability theory) satisfies similar algebraic properties as the Peirce projections on a ring R or on an algebra A. A result of J. Tomiyama states that a unital and bounded projection $E : A \to A$ with range $S = E(A)$ a C*-subalgebra of A must have norm one, must be positive, must satisfy the conditional expectation property $E(s_1xs_2) = s_1E(x)s_2$ (for $s_1, s_2 \in S, x \in A$) and also the Schwarz type inequality $E(x)^2 \leq E(x^2)$ for self-adjoint x (see [1, II.6.10.2]). In one of the themes of recent research, the notion of injective operator space, a similar algebraic 'conditional expectation' property plays a significant role, interacting with the notion of a ternary ring of operators (TRO, see [11]).

In [7], T. Y. Lam proposed abstracting the algebraic properties of the Peirce projection $E_e : R \to R$ associated with an idempotent e in a ring R, which is given by $E_e(x) = exe$, $(x \in R)$, and investigating algebraic properties that hold in this more general context. His proposal is to consider (additive) maps $E : R \to R$ with $E \circ E = E$, $S = E(R)$ a subring of R under the assumption that E is an S-bimodule map (which means that it satisfies the conditional expectation property $E(s_1xs_2) = s_1E(x)s_2$ for $s_1, s_2 \in S, x \in R$). Lam refers to such subrings S as 'corners'.

We consider this notion principally in the context of a (complex) C*-algebra A in place of a ring R and with the assumption that the corner $S = E(A)$ is a complex subalgebra. Our aim is to characterize such corners as fully as we can, ideally by establishing that they are related to the ranges of the more well-known completely positive (unital) conditional expectations.

In the general approach of Lam (in the context of rings), although a ring-theoretic Lam corner S of a unital algebra A need not be a subalgebra, if S is a subalgebra then the corresponding projection E must be linear (that is, homogeneous), which justifies the definition of corner algebra we use (Definition 2.1). Thus we adopt a definition modified from the ring-theoretic one (which insists that we deal with corners that are subalgebras and have vector space complements, or equivalently we deal only with linear projections E).

While simple examples show that Lam corners S in C*-algebras need not be self-adjoint subalgebras, Peirce corners in C*-algebras and certain 'generalized' Peirce corners behave like self-adjoint corners (see [10, section 3.6]). In Proposition 2.5, we characterize corners in finite dimensional C*-algebras that contain the diagonal and use that in Theorem 1 to characterize corners of von Neumann algebras that contain the diagonal in some basis for H. A consequence of this result is a version where the range of the projection on $\mathcal{B}(H)$ is a weakly closed ternary ring of operators (Theorem 2).